How to Find Time to Convergence

Three methods for solving this problem:

1) Algebraic Solution Method

We start with the following formula from the Lecture 1 Slides:

$$\frac{\text{Country 1 GDP after N years}}{\text{Country 2 GDP after N years}} = \frac{\text{Country 1 Total Growth after N Years}+100}{\text{Country 2 Total Growth after N Years}+100} \times \frac{\text{Country 1 GDP now}}{\text{Country 2 GDP now}}$$

Where the formula for total growth after N years for a given growth rate, r, is

$$\text{Total Growth after N Years} = 100 \times \left(\left[\frac{100+r}{100}\right]^N - 1\right)$$

**Note that by GDP we mean GDP per capita (since we're interested in convergence of income), but the formula works for all compound growth rates, e.g. GDP, Population, stock market portfolios

First, we want the GDP in the two countries to be equal after N years, so the left hand side will equal 1.

$$1 = \frac{\text{Country 1 Total Growth after N Years}+100}{\text{Country 2 Total Growth after N Years}+100} \times \frac{\text{Country 1 GDP now}}{\text{Country 2 GDP now}}$$

Multiplying both sides by $\frac{\text{Country 2 GDP now}}{\text{Country 1 GDP now}}$

$$\frac{\text{Country 2 GDP now}}{\text{Country 1 GDP now}} \times 1 = \frac{\text{Country 1 Total Growth after N Years}+100}{\text{Country 2 Total Growth after N Years}+100} \times \frac{\text{Country 1 GDP now}}{\text{Country 2 GDP now}} \times \frac{\text{Country 2 GDP now}}{\text{Country 1 GDP now}}$$

Cancelling terms and simplifying gives

$$\frac{\text{Country 2 GDP now}}{\text{Country 1 GDP now}} = \frac{\text{Country 1 Total Growth after N Years}+100}{\text{Country 2 Total Growth after N Years}+100}$$

Let r_1 be the growth rate of country 1 and r_2 be the growth rate of country 2.

Therefore

$$\text{Country 1 Total Growth after N Years}+100 =$$

$$= 100 \times \left(\left[\frac{100+r_1}{100}\right]^N - 1\right) + 100$$

$$= 100 \times \left[\frac{100+r_1}{100}\right]^N - 100 + 100$$

$$= 100 \times \left[\frac{100 + r_1}{100}\right]^N$$

And similarly for Country 2 we have

$$\text{Country 2 Total Growth after N Years+100} = 100 \times \left[\frac{100 + r_2}{100}\right]^N$$

Note that due to the properties of multiplication: $\left[\frac{100+r}{100}\right]^N = [100 + r]^N \times \left[\frac{1}{100}\right]^N$

Therefore combining the two equations we have

$$\frac{\text{Country 1 Total Growth after N Years+100}}{\text{Country 2 Total Growth after N Years+100}} = \frac{100 \times \left[\frac{100 + r_1}{100}\right]^N}{100 \times \left[\frac{100 + r_2}{100}\right]^N}$$

$$= \frac{100 \times [100 + r_1]^N \times \left[\frac{1}{100}\right]^N}{100 \times [100 + r_2]^N \times \left[\frac{1}{100}\right]^N}$$

Cancelling terms

$$= \frac{[100 + r_1]^N}{[100 + r_2]^N}$$

And again exploiting the properties of multiplication

$$= \left[\frac{100 + r_1}{100 + r_2}\right]^N$$

Therefore plugging in that $\frac{\text{Country 1 Total Growth after N Years+100}}{\text{Country 2 Total Growth after N Years+100}} = \left[\frac{100+r_1}{100+r_2}\right]^N$, our expression

$$\frac{\text{Country 2 GDP now}}{\text{Country 1 GDP now}} = \frac{\text{Country 1 Total Growth after N Years+100}}{\text{Country 2 Total Growth after N Years+100}}$$

becomes

$$\frac{\text{Country 2 GDP now}}{\text{Country 1 GDP now}} = \left[\frac{100 + r_1}{100 + r_2}\right]^N$$

We can solve this problem taking logarithms since $\log X^a = a \log X$ for any two numbers X and a, where X must be positive.

Taking the log of both sides gives

$$\log\left(\frac{\text{Country 2 GDP now}}{\text{Country 1 GDP now}}\right) = N\log\left(\frac{100+r_1}{100+r_2}\right)$$

And dividing both sides by $\log\left(\frac{100+r_1}{100+r_2}\right)$ and flipping the sides of the equation gives

$$N = \frac{\log\left(\frac{\text{Country 2 GDP now}}{\text{Country 1 GDP now}}\right)}{\log\left(\frac{100+r_1}{100+r_2}\right)}$$

Which gives us a formula we can use to find our answer. We just need to plug in the values on the RHS and it will tell us the number of years N after which the two countries converge.

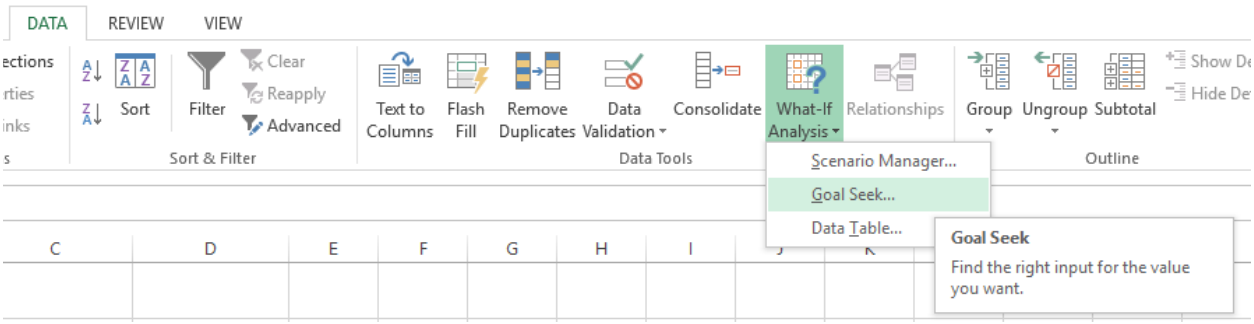
2) Trial-and-Error

Plug different values of N into the formulas at the top of lecture 1. Keep trying until you find one that works.

2) Numerical Solver

Put the formulas on the computer and allow them to update automatically as you change your guess for N. Compute the difference between the answer you get and the answer you want to get, and run a numerical root finder to find the value of N that sets the difference equal to zero.

In excel, you can do this using goal seek:



Note that goal seek may not set the difference equal to exactly zero. It runs an algorithm that tries to get close to zero and stops when it decides it is close enough (referred to as the tolerance). Therefore the solution it gives is any

The Solver add-on for excel lets you choose the tolerance yourself for better approximations.