ECO 745 Notes: Deriving DFS (1979) equilibrium for a symmetric world

Economy Setup

•There are two countries: $i, j = 1,2$

•A continuum of goods: $z \in [0,1]$

•Labor is the only factor of production, and is supplied inelastically. $L_1 = L_2 = L$

•Cobb Douglas preferences over goods

• Countries differ only in labor productivities for each good

$$y_j(z) = l_j(z)/a_j(z)$$

$$a_1(z) = e^{az}$$

$$a_2(z) = e^{a(1-z)}$$

• No trade costs (this implies price of goods will be same in both countries)

Equilibrium Definition:

An equilibrium in this economy is prices $\{\hat{p}(z)\}_{z\in[0,1]}$, wages $\widehat{w}_1, \widehat{w}_2$, consumption $\{\hat{c}_j(z)\}_{z\in[0,1]}$, output $\{\hat{y}_j(z)\}_{z\in[0,1]}$, and labor allocation $\{\hat{l}_j(z)\}_{z\in[0,1]}$ such that

1. Consumers Problem

Given $\hat{p}(z), \widehat{w}_j$ consumers in country j chooses consumption to maximize utility

$$\max \int_0^1 \log\left(c_j(z)\right)\, dz$$

Subject to their budget constraint and non-negative consumption

$$\int_0^1 \hat{p}(z) c_j(z) = \widehat{w}_j L_j$$

$$c_j(z) \geq 0,\ \forall z$$

2. Firms Problem

Given $\hat{p}(z), \widehat{w}_j$, firms in country j chooses output and labor input to maximize profits for each good z:

$$\max \hat{p}(z) y_j(z) - \widehat{w}_j l_j(z)$$

Subject to the technology constraint

$$y_j(z) = \frac{l_j(z)}{a_j(z)}$$

3. Market Clearing

The goods market clears:

$$\hat{c}_1(z) + \hat{c}_2(z) = \hat{y}_1(z) + \hat{y}_2(z), \qquad \forall z \in [0,1]$$

The labor market clears:

$$\int_0^1 \hat{l}_j(z) = L_j, \qquad j = 1,2$$

Solving Equilibrium

From consumers problem we can set up the Lagrangian

$$\mathcal{L} = \int_0^1 \log c_j(z)\, dz + \lambda_j \left(w_j L - \int_0^1 \hat{p}(z) c_j(z)\, dz \right)$$

And take the FOC with respect to $c_j(z)$ ($[X]$: is personal notation for taking derivative with respect to X)

$$[c_j(z)]:\ \frac{1}{c_j(z)} - \lambda_j \hat{p}(z) = 0$$

Using that FOC twice for two separate goods, z and z', then taking the ratio yields:

$$\left(\frac{1}{c_j(z)}\right) / \left(\frac{1}{c_j(z')}\right) = \left(\lambda_j \hat{p}(z)\right) / \left(\lambda_j \hat{p}(z')\right)$$

$$c_j(z) = \frac{\hat{p}(z') c_j(z')}{\hat{p}(z)}$$

We can do that for all goods $z \in [0,1]$, and then substitute those formulas in the budget constraint to get:

$$\int_0^1 \hat{p}(z) \frac{\hat{p}(z') c_j(z')}{\hat{p}(z)} dz = \widehat{w}_j L_j$$

And simplifying we get

$$\hat{p}(z') c_j(z') \int_0^1 1\, dz = \widehat{w}_j L_j$$

And that holds for any arbitrary good z', therefore we have $\forall z \in [0,1]$:

$$\hat{c}_j(z) = \frac{\widehat{w}_j L_j}{\hat{p}(z)}, \qquad \mathbf{(1)}$$

Using the Firm's Problem

Now we can solve the firm's problem for each country. Rearranging the technology function we have

$$l_j(z) = a_j(z)y_j(z) = \begin{cases} e^{az}, & if\ j = 1 \\ e^{a(1-z)}, & if\ j = 2 \end{cases}$$

And plugging that into the firm maximization problem gives:

$$\max \hat{p}(z)y_j(z) - w_j\left(a_j(z)y_j(z)\right)$$

Taking the FOC wrt $y_1(z)$ for country 1 gives:

$$[y_1(z)]: \hat{p}(z) - a_1(z)w_1 \leq 0;\ \ w.e.\,if\ y_1(z) > 0$$

And taking the FOC wrt $y_2(z)$ for country 2 gives:

$$[y_2(z)]: \hat{p}(z) - a_2(z)w_2 \leq 0;\ \ w.e.\,if\ y_2(z) > 0$$

And firms only produce if they can earn non-negative profits, therefore prices will be

$$\hat{p}(z) = min\{w_1 a_1(z), w_2 a_2(z)\}$$

Or, plugging in labor productivities:

$$\boldsymbol{\hat{p}(z) = min\left\{w_1 e^{az}, w_2 e^{a(1-z)}\right\}}, \qquad (\mathbf{2})$$

Now, we know there will be a cutoff good $\bar{z}$ such that Z_j is the set of goods that country j produces, where:

$$Z_1 = [0, \bar{z}]$$

$$Z_2 = [\bar{z}, 1]$$

Note at z' the cost of producing the good will be the same in both countries, therefore:

$$w_1 e^{a\bar{z}} = w_2 e^{a(1-\bar{z})}$$

Which implies that relative wages must satisfy:

$$\boldsymbol{\frac{w_1}{w_2} = e^{a-2a\bar{z}}}, \qquad (\mathbf{3})$$

Using Market Clearing

Going back to market clearing conditions, we have

$$\hat{c}_1(z) + \hat{c}_2(z) = \hat{y}_1(z) + \hat{y}_2(z)$$

And every good (except possibly good z') is only produced by one country, we have:

$$\hat{c}_1(z) + \hat{c}_2(z) = \hat{y}_j(z), \qquad if\ z \in Z_j$$

Multiplying both sides by $\hat{p}(z)$:

$$\hat{p}(z)\hat{c}_1(z) + \hat{p}(z)\hat{c}_2(z) = \hat{p}(z)\hat{y}_j(z), \qquad if\ z \in Z_j$$

Plugging in equation **(1)** from our consumers problem for both countries yields:

$$\widehat{w}_1 L_1 + \widehat{w}_2 L_2 = \hat{p}(z)\hat{y}_j(z), \qquad if\ z \in Z_j \qquad (4)$$

From 2 and the production function we have that

$$\hat{p}(z)\hat{y}_j(z) = \widehat{w}_j a_j(z)\hat{y}_j(z) = \widehat{w}_j a_j(z)\frac{\hat{l}_j(z)}{a_j(z)} = \widehat{w}_j \hat{l}_j(z), \qquad if\ z \in Z_j$$

Therefore plugging the above into (4) yields:

$$\widehat{w}_1 L_1 + \widehat{w}_2 L_2 = \widehat{w}_j \hat{l}_j(z), \qquad if\ z \in Z_j$$

Now for each country we can integrate over what it produces:

$$\int_{z\in Z_j} (\widehat{w}_1 L_1 + \widehat{w}_2 L_2) dz = \int_{z\in Z_j} \widehat{w}_j \hat{l}_j(z) dz$$

Pulling the constants out of the integrals:

$$(\widehat{w}_1 L_1 + \widehat{w}_2 L_2) \int_{z\in Z_j} dz = \widehat{w}_j \int_{z\in Z_j} \hat{l}_j(z) dz$$

Plugging in labor market clearing:

$$(\widehat{w}_1 L_1 + \widehat{w}_2 L_2) \int_{z\in Z_j} dz = \widehat{w}_j L_j, \qquad (5)$$

And now note that, since $z \in [0,1]$, we have:

$$\int_{z\in Z_j} dz = \begin{cases} \bar{z}, & if\ j = 1 \\ (1-\bar{z}), & if\ j = 2 \end{cases}$$

Therefore plugging that into (5) we have

$$\bar{z}(\widehat{w}_1 L_1 + \widehat{w}_2 L_2) = \widehat{w}_1 L_1$$

$$(1-\bar{z})(\widehat{w}_1 L_1 + \widehat{w}_2 L_2) = \widehat{w}_2 L_2$$

Therefore using the above two equations and rearranging we get:

$$\boldsymbol{\frac{\widehat{w}_1}{\widehat{w}_2} = \frac{\bar{z}}{1-\bar{z}}\frac{L_2}{L_1}}, \qquad \mathbf{(6)}$$

Now combining (6) with (3), and plugging in $L_1 = L_2 = L$ yields:

$$\frac{\bar{z}}{1-\bar{z}} = e^{a-2a\bar{z}}$$

Which means that $\bar{z} = \frac{1}{2}$. We could have exploited symmetry to arrive at this conclusion immediately, but here we learned how to derive it without relying on symmetry.

Equilibrium Allocations

We can normalize $\hat{w}_1 = 1$, and since $\bar{z} = \frac{1}{2}$ that implies that $\frac{\hat{w}_1}{\hat{w}_2} = 1$ and therefore $\hat{w}_2 = 1$

Plugging wages into (2) yields prices:

$$\hat{p}(z) = \begin{cases} e^{az}, & z \in \left[0, \frac{1}{2}\right] \\ e^{a(1-z)}, & z \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Which we can combine with (1) to get quantities consumed:

$$\hat{c}_j(z) = \begin{cases} \frac{L}{e^{az}}, & z \in [0, .5\ ] \\ \frac{L}{e^{a(1-z)}}, & z \in [.5,0] \end{cases}, \qquad j = 1,2$$

And then goods market clearing implies:

$$\hat{y}_1(z) = \begin{cases} \frac{2L}{e^{az}}, & z \in [0, .5\ ] \\ 0, & z \in [.5,0] \end{cases}$$

$$\hat{y}_2(z) = \begin{cases} 0, & z \in [0, .5\ ] \\ \frac{2L}{e^{a(1-z)}}, & z \in [.5,0] \end{cases}$$

And using the production technology gives:

$$\hat{l}_1(z) = \begin{cases} 2L, & z \in [0, .5\ ] \\ 0, & z \in [.5,0] \end{cases}$$

$$\hat{l}_2(z) = \begin{cases} 0, & z \in [0, .5\ ] \\ 2L, & z \in [.5,0] \end{cases}$$

DFS With Tariffs

Now suppose that each country imposes an ad valorem tariff $\tau - 1$ on imports from the other country.

Equilibrium Definition (sloppier with notation this time)

An equilibrium for this economy is prices $\{\hat{p}_j(z), \widehat{w}_j\}_{j=1,2;z\in[0,1]}$ and allocations $\{\hat{c}_j(z), \hat{y}_j(z), \hat{l}_j(z)\}_{j=1,2;z\in[0,1]}$ such that **1, 2, 3** hold below

1. Given prices the consumer solves:

$$\max \int_0^1 \log c_j(z)\, dz$$

subject to their budget constraint and non-negative consumption (T_j is revenue from tariffs)

$$\int_0^1 p_j(z) c_j(z)\, dz = w_j L + T_j$$

$$c_j(z) \geq 0 \ \ \forall z \in [0,1]$$

2. Firms maximize profits in each country $j = 1,2$ for each good z:

$$\max p_j^i(z) y_j^i(z) - w_j l_j^i(z)$$

Where (y_j^i is output produced in country j for consumption in country i, and $\tau - 1$ is the tariff)

$$y_j^i(z) = \frac{l_j^i(z)}{a_j(z)}$$

And countries have to charge a tariff on their exports to the other country

$$p_j^i(z) = \tau p_j^j, \qquad i \neq j$$

3. Markets clear, $\forall z \in [0,1];\ j = 1,2$:

$$c_j(z) = y_1^j(z) + y_2^j(z)$$

$$\int_0^1 \left(l_j^1(z) + l_j^2(z) \right) = L$$

And tariffs are transferred back to the consumer through T_j, where T_j is defined as:

$$T_j = \int_{[0,1]\backslash Z_j} (\tau - 1) p_i(z) y_i^j(z)\, dz\,, \qquad i \neq j$$

Here Z_j is the set of goods country j produces (and so $[0,1]\backslash Z_j$ is the set they import)

Solving for the Equilibrium

Assume that tariffs aren't so high that the countries stop trading with each other: $\tau \leq \min\left\{\frac{w_2}{w_1} e^a, \frac{w_1}{w_2} e^a\right\}$

Now we know there will be two cutoff goods: $\bar{z}_1, \bar{z}_2; \bar{z}_1 \le \bar{z}_2$ such that the set of goods produced by each country is:

$$Z_1 = [0, \bar{z}_2]$$

$$Z_2 = [\bar{z}_1, 1]$$

And the set of goods exported by each country is

$$Z_1^2 = [0, \bar{z}_1]$$

$$Z_2^1 = [\bar{z}_2, 1]$$

Solving Consumers Problem:

The consumer's problem is essentially the same, except now he gets revenues from tariffs therefore something similar to equation (1) holds:

$$c_j(z) = \frac{w_j L \ + T_j}{p_j(z)} \ \ \forall z \in [0,1], \qquad (7)$$

Solving Firms Problem:

The firm's problem is the same as in the free trade model, except now τ is applied to the price for exports. Therefore from the FOC for the firms problem we have prices given by

$$p_j(z) = \min\left\{p_i^j, p_j^j\right\}$$

Where

$$p_i^j = \begin{cases} w_i a_i(z), & i = j \\ \tau w_i a_i(z), & i \neq j \end{cases}$$

Production and Exporting Cutoffs:

Now, instead of having one cutoff good where the effective cost of producing is equal in both countries as in eqn (3), we have two such cutoff goods:

$$\frac{w_1}{w_2} = \tau^{-1} e^{a-2a\bar{z}_2}, \qquad (8)$$

$$\frac{w_1}{w_2} = \tau e^{a-2a\bar{z}_1}, \qquad (9)$$

And trade must be balanced (exports=imports) which means that:

$$\int_0^{\bar{z}_1} p_1^2(z) c_2(z)\, dz = \int_{\bar{z}_2}^1 p_2^1(z) c_1(z) dz$$

And from (7) this means

$$\int_0^{\bar{z}_1} (w_2 L + T_2)\, dz = \int_{\bar{z}_2}^1 (w_1 L + T_1) dz$$

Therefore

$$(w_2L+T_2)\bar{z}_1=(w_1L+T_1)(1-\bar{z}_2)$$

And we can normalize $w_2=1$, giving us:

$$w_1=\frac{L\bar{z}_1+T_2\bar{z}_1-T_1+\bar{z}_2T_1}{L(1-\bar{z}_2)}$$

Combining the above with equations (8) and (9) gives us three equations in three unknowns $(w_1,\bar{z}_1,\bar{z}_2)$.

Solving these three equations yields:

$$\bar{z}_1=\frac{1}{2}\left(1-\frac{1}{a}\log\tau\right)$$

$$\bar{z}_2=1-\frac{1}{2}\left(1-\frac{1}{a}\log\tau\right)$$

and $w_1=w_2=1$.

Equilibrium Allocations:

Plugging in our wages to get prices, means that equilibrium allocations will be given by

$$c_1(z)=\begin{cases}\dfrac{L+T_1}{e^{az}}, & z\in[0,\bar{z}_2]\\ \dfrac{L+T_1}{\tau e^{a(1-z)}}, & z\in(\bar{z}_2,1]\end{cases}$$

$$c_2(z)=\begin{cases}\dfrac{L+T_2}{\tau e^{az}}, & z\in[0,\bar{z}_1]\\ \dfrac{L+T_2}{e^{a(1-z)}}, & z\in(\bar{z}_1,1]\end{cases}$$

where

$$p_1(z)=\begin{cases}e^{az}, & z\in[0,\bar{z}_2]\\ \tau e^{a(1-z)}, & z\in(\bar{z}_2,1]\end{cases}$$

$$p_2(z)=\begin{cases}\tau e^{az}, & z\in[0,\bar{z}_1]\\ e^{a(1-z)}, & z\in(\bar{z}_1,1]\end{cases}$$

And tariff transfers can be computed by, for country 1

$$T_1=\int_{[\bar{z}_2,1]}(\tau-1)p_2(z)y_2^1(z)dz=\int_{[\bar{z}_2,1]}(\tau-1)p_2(z)\left(\frac{L\ +T_1}{p_1(z)}\right)dz=\int_{[\bar{z}_2,1]}(\tau-1)p_2(z)\left(\frac{L\ +T_1}{\tau p_2(z)}\right)dz$$

$$=\int_{[\bar{z}_2,1]}\left(\frac{\tau-1}{\tau}\right)(L\ +T_1)dz=\left(\frac{\tau-1}{\tau}\right)(L\ +T_1)(1-\bar{z}_2)$$

which yields

$$T_1 = \frac{L\left(\frac{\tau-1}{\tau}\right)(1-\bar{z}_2)}{1-\left(\frac{\tau-1}{\tau}\right)(1-\bar{z}_2)}$$

And similarly, for country 2

$$T_2 = \int_{[0,\bar{z}_1]} \left(\frac{\tau-1}{\tau}\right)(L+T_2)dz \Rightarrow T_2 = \frac{L\left(\frac{\tau-1}{\tau}\right)\bar{z}_1}{1-\left(\frac{\tau-1}{\tau}\right)\bar{z}_1}$$

And note that $\bar{z}_1 = 1-\bar{z}_2$, therefore $T_1 = T_2 = T$, as expected in a symmetric world. Plugging in cutoffs we have:

$$T = \frac{L(\tau-1)(a-\log\tau)}{a(1+\tau)+(\tau-1)\log\tau}$$

Note that therefore total income is

$$L+T = \frac{L}{1-\left(\frac{\tau-1}{\tau}\right)\bar{z}_1} = \frac{2aL\tau}{a(1+\tau)+(\tau-1)\log\tau}$$

Note also that we can compute GDP as

$$GDP_1 = \int_0^{\bar{z}_2} p_1(z)c_1(z)dz + \int_0^{\bar{z}_1} p_1^2(z)c_2(z)dz$$

$$\int_0^{\bar{z}_2}(L_1+T_1)\,dz + \int_0^{\bar{z}_1}(L_2+T_2)dz = L+T = \frac{2aL\tau}{a(1+\tau)+(\tau-1)\log\tau}$$

We can also compute Constant Price GDP, for a base period in which $\tau = 1$, as

$$RGDP_1 = \int_0^{\bar{z}_2} \tilde{p}_1(z)c_1(z)dz + \int_0^{\bar{z}_1} \tilde{p}_1^2(z)c_2(z)dz$$

where the $\tilde{p}(z)$ are the prices from the base case of free trade.

And note that $\tilde{p}_1 = p_1$ for $z \in \left[0,\frac{1}{2}\right]$, and $\tilde{p}_1 = e^{a(1-z)} = \frac{e^{a(1-z)}}{e^{az}}p_1$, and $\tilde{p}_1^2 = p_1^2/\tau$ therefore

$$RGDP_1 = \int_0^{\frac{1}{2}} p_1(z)c_1(z)dz + \int_{\frac{1}{2}}^{\bar{z}_2} \frac{e^{a(1-z)}}{e^{az}} p_1(z)c_1(z)dz + \int_0^{\bar{z}_1} \frac{1}{\tau} p_1^2(z)c_2(z)dz$$

And plugging in that $p_1(z)c_1(z) = L+T\ \forall z \in [0,\bar{z}_2]$ and $p_1^2(z)c_2(z) = L+T\ \forall z \in [0,\bar{z}_1]$:

$$RGDP_1 = \int_0^{\frac{1}{2}} (L+T)dz + \int_{\frac{1}{2}}^{\bar{z}_2} \frac{e^{a(1-z)}}{e^{az}} (L+T)dz + \int_0^{\bar{z}_1} \frac{1}{\tau} (L+T)dz$$

$$= (L+T) \int_0^{\frac{1}{2}} dz + (L+T) \int_{\frac{1}{2}}^{\bar{z}_2} \frac{e^{a(1-z)}}{e^{az}} dz + \frac{1}{\tau} (L+T) \int_0^{\bar{z}_1} dz$$

$$= (L+T)\frac{1}{2} + (L+T)\left(\frac{1-e^{a-2a\bar{z}_2}}{2a}\right) + \frac{1}{\tau}(L+T)\bar{z}_1$$

$$= GDP_1\left(\frac{1}{2} + \left(\frac{1-e^{a-2a\bar{z}_2}}{2a}\right) + \frac{1}{\tau}\bar{z}_1\right)$$

Plugging in cutoffs

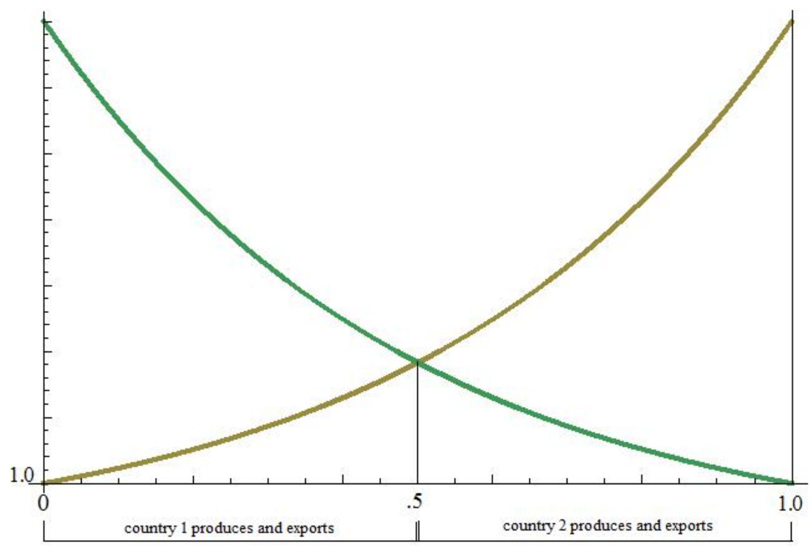
$$RGDP_1 = GDP_1\left(\frac{1}{2} + \left(\frac{1-e^{a-2a\left(1-\frac{1}{2}\left(1-\frac{1}{a}\log\tau\right)\right)}}{2a}\right) + \frac{1}{\tau}\frac{1}{2}\left(1-\frac{1}{a}\log\tau\right)\right)$$

$$= GDP_1\left(\frac{1}{2} + \frac{1}{2}\left(\frac{\tau-1}{a\tau}\right) + \frac{1}{\tau}\frac{1}{2}\left(1-\frac{1}{a}\log\tau\right)\right)$$

$$= \frac{GDP_1}{2}\left(\frac{a+\tau-1+a\tau-\log\tau}{a\tau}\right)$$

Which equals current price GDP if $\tau = 1$ and is less than current price GDP otherwise.

Free trade:



Tariffs/Iceberg Costs: