ECO 745: Theory of International Economics

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Monopolistic Competition with Homogeneous Firms

Krugman (1980) framework

- Countries are identical, but firms produce differentiated products
- All firms have the same technology
- Countries trade because consumers like variety / CES preferences

Trade costs do not affect the number of firms that enter in each country

- Higher trade costs do lower welfare and trade flows.
- A higher elasticity of substitution between products amplifies the impact of trade barriers
    - Trade costs are reflected in prices and consumers will substitute away from foreign goods more

Monopolistic Competition with Heterogeneous Firms

Chaney (2008) shows how results change when firms are no longer homogeneous. Setup:

- Countries are identical, but firms produce differentiated products
- Firms differ in their technology. Some firms are more productive than others.
- Countries trade because consumers like variety / CES preferences.

Results: Trade costs will affect both the number of firms that enter in each country (extensive margin) and firm-level trade flows (intensive margin)

- A higher elasticity of substitution dampens the impact of trade barriers in this environment
- Today will discuss how to solve model. Next time will discuss this result.

CES with Heterogeneous Firms: Setup

Chaney (2008) builds off Melitz (2003).

- $i, j = 1, \ldots, N$ Countries
- Fixed mass of differentiated firms in each country, which differ in their productivity levels
    - Firm productivity distribution will follow a Pareto distribution
- Consumers have CES preferences over differentiated goods, and also consume a homogeneous good.
- Firms pay a fixed cost to enter and produce differentiated goods
- Differentiated firms pay a fixed cost to export, and are subject to iceburg trade costs
    - Homogeneous good is freely traded, so price is equalized across countries

Household Demand

Household side is same as Krugman (1980). Consumers in country j solve

$$\max_{\{c_{j,0}, c_j(m)\}} (1-\theta)\log c_{j,0} + \frac{\theta}{\rho}\log\left(\int_{m\in M_j} \left(c_j(m)\right)^{\rho} dm\right)$$

subject to budget constraint:

$$p_{j,0}c_{j,0} + \int_{m\in M_j} p_j(m)c_j(m) = w_jL_j + \pi_j$$

where $w_jL_j + \pi_j = I_j$ is the income of households in country j, and M_j is the set of firms that produce output for consumers in country j.

Household Demand

After taking FOC and solving for demand, we have

$$c_{j,0} = \frac{(1-\theta)\left(w_j L_j + \pi_j\right)}{p_{j,0}}$$

$$c_j(k) = \frac{\theta\left(w_j L_j + \pi_j\right)}{\left(p_j(k)\right)^{\frac{1}{1-\rho}} \left(P_j\right)^{-\left(\frac{\rho}{1-\rho}\right)}}, k \in M_j$$

where P_j is the CES price index

$$P_j = \left(\int_{m \in M_j} \left(p_j(m)\right)^{-\left(\frac{\rho}{1-\rho}\right)} dm\right)^{-\left(\frac{1-\rho}{\rho}\right)}$$

Homogeneous Good Problem

Perfectly competitive firms produce homogeneous good according to

$$y_{j,0} = l_{j,0}$$

Therefore, in equilibrium the price of the homogeneous good will be equal to the wage

$$p_{j,0} = w_j$$

The homogeneous good is freely traded, therefore wages will be equal across countries

$$p_{j,0} = p_{i,0} \Rightarrow w_j = w_i, \qquad \forall i, j$$

Can normalize wages equal to 1 (won't do it right away, but will later).

Differentiated Firm Setup

First consider Autarky:

- Fixed mass of firms μ, each firm produces a differentiated good
- Firm m has productivity $z(m)$. Distribution of $z(m)$ follows a Pareto distribution.
- Firms must pay a fixed cost, f, in terms of labor in order to produce
  - Firms know their productivity prior to paying fixed cost

Firm Maximization Problem

Taking demand as given, firm m maximizes profits

$$\max_{p(m)} p(m)y(m) - wl(m)$$

Subject to production function

$$y(m) = \max\{z(m)(l(m) - f), 0\}$$

Note that if firm m produces then $l(m) = y(m)/z(m) + f$

Solving for Firm Prices: Same as Before

Substituting demand into the firm maximization problem makes it:

$$\max_{p_j(m)} \left(p_j(m)\right)^{\frac{-\rho}{1-\rho}} \frac{\theta\left(w_j L_j + \pi_j\right)}{\left(P_j\right)^{-\left(\frac{\rho}{1-\rho}\right)}} - \left(p_j(m)\right)^{\frac{-1}{1-\rho}} \frac{w_j}{z_j(m)} \frac{\theta\left(w_j L_j + \pi_j\right)}{\left(P_j\right)^{-\left(\frac{\rho}{1-\rho}\right)}} - w_j f$$

To solve for prices take FOC of the above. If $y_j(m) > 0$, then

$$\left[p_j(m)\right]: \left(-\frac{\rho}{1-\rho}\right)\left(p_j(m)\right)^{\frac{-\rho}{1-\rho}-1} \frac{\theta\left(w_j L_j + \pi_j\right)}{\left(P_j\right)^{\frac{-\rho}{1-\rho}}} - \left(\frac{-1}{1-\rho}\right)\left(p_j(m)\right)^{\frac{-1}{1-\rho}-1}\left(\frac{w_j}{z_j(m)}\right) \frac{\theta\left(w_j L_j + \pi_j\right)}{\left(P_j\right)^{-\left(\frac{\rho}{1-\rho}\right)}} = 0$$

Simplifying yields:

$$p_j(m) = \left(\frac{w_j}{\rho z_j(m)}\right)$$

Firm Prices and Profits

Price is a markup over marginal cost, same as before, but now marginal cost is firm specific

$$p_j(m) = \overbrace{\rho^{-1}}^{\text{markup}} \overbrace{\left(w_j / z_j(m)\right)}^{\text{marginal cost}}$$

Firm profits given by same formula as before, and firms will produce if $\pi_j(m) > 0$:

$$\pi_j(m) = p_j(m) y_j(m) - w_j l_j(m) = \left(\frac{w_j}{\rho z_j(m)} - \frac{w_j}{z_j(m)}\right)\left(\frac{\theta(w_j L_j + \pi_j)}{\left(\frac{w_j}{\rho z_j(m)}\right)^{\frac{1}{1-\rho}} (P_j)^{-\left(\frac{\rho}{1-\rho}\right)}}\right) - w_j f$$

However, to get price index need to know distribution of $z_j(m)$'s:

$$P_j = \left(\int_{m \in M_j} \left(\frac{w_j}{\rho z_j(m)}\right)^{-\left(\frac{\rho}{1-\rho}\right)} dm\right)^{-\left(\frac{1-\rho}{\rho}\right)}$$

Firm Productivity Distribution

Assume that firm productivities follow a Pareto distribution with cutoff 1 and tail parameter $\gamma > 2$

Therefore the cumulative distribution function for productivity will be

$$F(z) \coloneqq P(Z \leq z) = \begin{cases} 1 - z^{-\gamma}, & z \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

And the probability density function will be, for $z \geq 1$,

$$dF(z) = \gamma z^{-\gamma-1}$$

Pareto Distribution with Different Cutoffs

Suppose X follows a Pareto distribution with cutoff k and tail parameter γ

The cumulative distribution function for a Pareto(k, γ) distribution will be

$$F(z) \coloneqq P(Z \leq z) = \begin{cases} 1 - (z/k)^{-\gamma}, & z \geq k \\ 0, & \text{otherwise} \end{cases}$$

And the probability density function will be, for $z \geq k$,

$$dF(z) = \gamma k^{\gamma} z^{-\gamma-1}$$

Properties of Pareto Distributions

- Mean of Pareto distribution is $E[X] = \left(\frac{\gamma}{\gamma-1}\right)k$ if $\gamma > 1$ (otherwise $E[X] = \infty$)

- Variance of Pareto distribution is $Var[X] = \frac{\gamma}{(\gamma-2)(\gamma-1)^2}k^2$ if $\gamma > 2$ (otherwise $Var[X] = \infty$)

- If we truncate a Pareto(k, γ) from below at $k' > k$, then the truncated random variable follows a Pareto(k', γ) distribution as $P(X > x | X > k') = (x/k')^{-\gamma}$ for $x \geq k'$ and = 1 otherwise.

- If X~Pareto(k, γ) then bX^a~Pareto$\left((bk)^a, \gamma/a\right)$

- If X~Pareto(k, γ) then $\log\left(1 - F(x)\right) = \gamma \log k - \gamma \log x$, i.e. the log of the CCDF is linear in log

Rank Analysis

In general, if n iid samples are drawn from a distribution then, for n large, the rank of a draw x is

$$rank(x) \approx n\big(1 - P(X < x)\big)$$

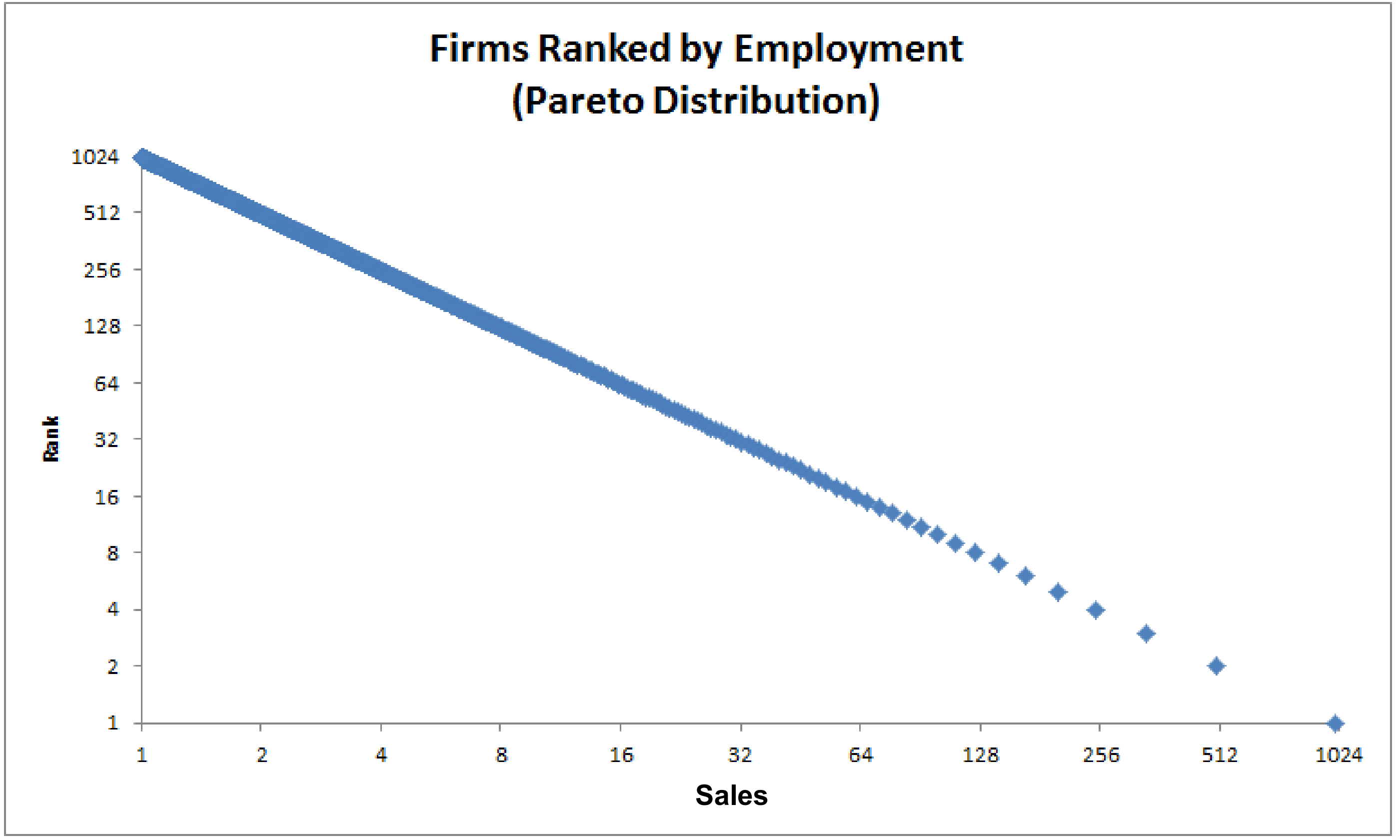
Therefore if the n draws are from a Pareto distribution with cutoff k and tail γ

$$\log\big(rank(x)\big) \approx \gamma \log k - \gamma \log x + \log n$$

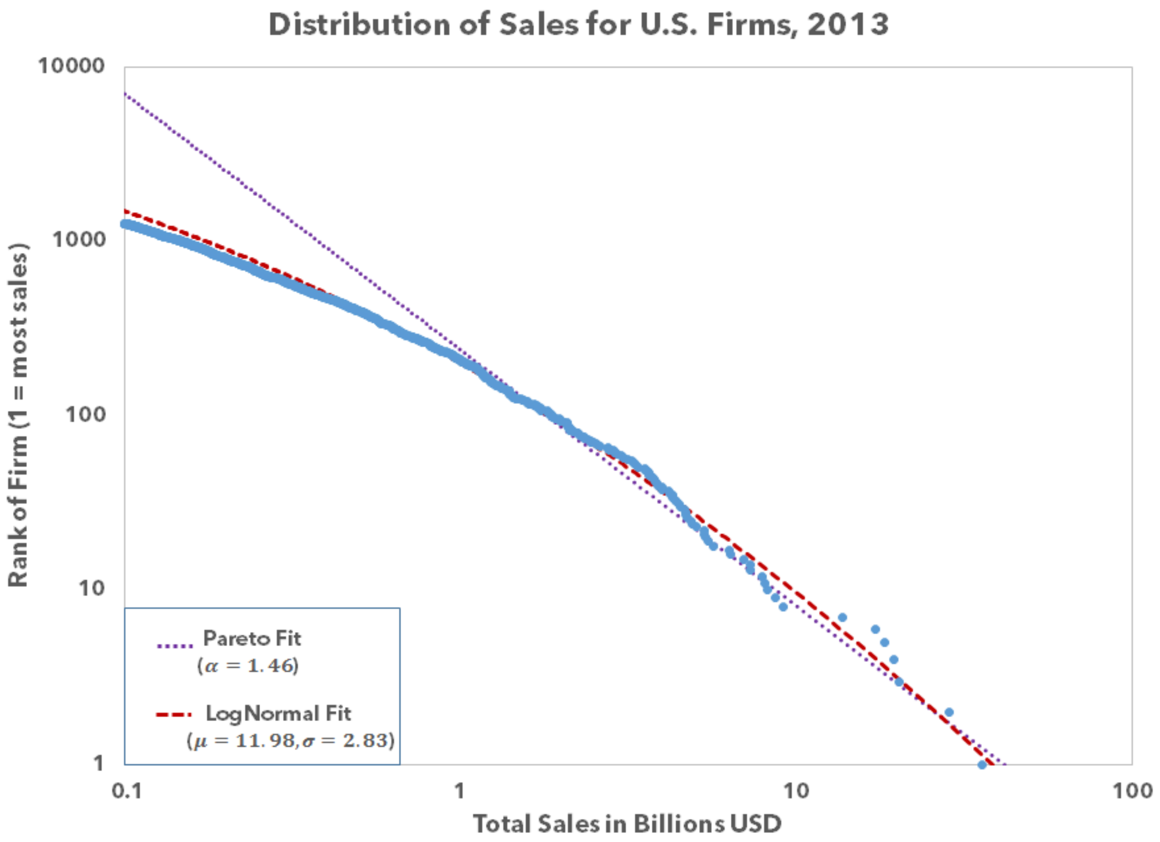
In the model, in equilibrium, $l(m)$ and $p(m)y(m)$ are proportional to $z(m)$. Therefore if $z(m)$ follows a Pareto distribution so should $l(m)$ and $p(m)y(m)$.

- Therefore, log rank in employment (sales) should be linear in log employment (log sales)

What Should a Pareto Distribution Look Like?



What Does Distribution of Sales in U.S. Tail Look Like?



Note of Caution on Firm Size Distribution

There is a debate over whether the distribution of firm sizes is Pareto or LogNormal

- Similar debate for city sizes, and in other fields such as computer sciences (e.g. website links)
- In practice, simply looking at the log-log plot is not sufficient to distinguish between them
- Instead should use Maximum Entropy methods or Uniformly Most Powerful Unbiased Test of Del Castillo and Puig (1999)

Also, traditional methods of estimating the Pareto tail parameter lead to biased estimates, but can correct for this fairly easily, e.g. Mette (1990).

Productivity, Fixed Costs, and Cutoffs: Autarky

Suppose there is a fixed mass of firms μ_j, and their productivity follows a Pareto distribution with cutoff 1 and tail parameter $\gamma > 2$.

Firms know their productivity before paying fixed cost to produce. There are two possible equilibrium cases depending on the level of fixed costs.

Case 1: All firms produce and earn positive profits

- Only happens if fixed costs are sufficiently low

Case 2: There is a cutoff productivity level, $\bar{z}$, such that firms at that cutoff productivity produce and earn zero profits, while firms above that cutoff productivity produce and earn positive profits, and firms below that cutoff productivity level do not produce.

- i.e. $y(m) > 0$ iff $z(m) \geq \bar{z}$

Price Index with Pareto Distribution for Productivity: Case 2

Firms are distinguished only by their productivity, so label firms by productivity from now on.

Assume we are in case 2, so there is a cutoff $\bar{z}$ such that firms only produce if $z > \bar{z}$

Rewrite price index as:

$$P_j = \left(\int_{m \in M_j} \left(\frac{w_j}{\rho z_j(m)} \right)^{-\left(\frac{\rho}{1-\rho}\right)} dm \right)^{-\left(\frac{1-\rho}{\rho}\right)} = \left(\mu_j \int_{\bar{z}}^{\infty} \left(\frac{w_j}{\rho z} \right)^{-\left(\frac{\rho}{1-\rho}\right)} dF(z) dz \right)^{-\left(\frac{1-\rho}{\rho}\right)}$$

Therefore

$$\left(P_j\right)^{-\left(\frac{\rho}{1-\rho}\right)} = \mu_j \int_{\bar{z}}^{\infty} \left(\frac{w_j}{\rho z} \right)^{-\left(\frac{\rho}{1-\rho}\right)} \left(\gamma z^{-\gamma-1}\right) dz$$

Price Index with Pareto Distribution for Productivity: Case 2

Computing the integral yields

$$\left(P_j\right)^{-\left(\frac{\rho}{1-\rho}\right)} = \mu_j \int_{\bar{z}}^{\infty} \left(\frac{w_j}{\rho z}\right)^{-\left(\frac{\rho}{1-\rho}\right)} \left(\gamma z^{-\gamma-1}\right) dz$$

$$= \gamma \mu_j \left(\frac{w_j}{\rho}\right)^{-\left(\frac{\rho}{1-\rho}\right)} \int_{\bar{z}}^{\infty} z^{(-\gamma-1)+\left(\rho/(1-\rho)\right)} dz$$

$$= \gamma \mu_j \left(\frac{w_j}{\rho}\right)^{-\left(\frac{\rho}{1-\rho}\right)} \left(\frac{1-\rho}{\rho-(1-\rho)\gamma}\right) \left[z^{\frac{\rho-(1-\rho)\gamma}{1-\rho}}\right]_{\bar{z}}^{\infty}$$

And here we require $\gamma > \rho/(1-\rho)$, in which case

$$\left(P_j\right)^{-\left(\frac{\rho}{1-\rho}\right)} = \gamma \mu_j \left(\frac{w_j}{\rho}\right)^{-\left(\frac{\rho}{1-\rho}\right)} \left(\frac{1-\rho}{(1-\rho)\gamma-\rho}\right) \bar{z}^{\frac{\rho-(1-\rho)\gamma}{1-\rho}}$$

Solving for Firm Output: Case 2 Autarky

Can substitute prices and price index into demand to get output for firm with productivity z. Dropping the subscript j and plugging in $w = 1$ (our normalization):

$$c(z) = \frac{\theta(L+\pi)}{p(z)^{\frac{1}{1-\rho}} P^{\frac{-\rho}{1-\rho}}} = \frac{\theta(L+\pi)}{\left(\frac{1}{\rho z}\right)^{\frac{1}{1-\rho}} \left(\mu\gamma\rho^{\frac{\rho}{1-\rho}} \left(\frac{1-\rho}{(1-\rho)\gamma-\rho}\right) \bar{z}^{\frac{\rho-(1-\rho)\gamma}{1-\rho}}\right)}$$

And simplifying the above yields

$$c(z) = \frac{\rho\big((1-\rho)\gamma-\rho\big)\theta(L+\pi)}{\mu\gamma(1-\rho)\bar{z}^{\frac{\rho-(1-\rho)\gamma}{1-\rho}}} z^{\frac{1}{1-\rho}}$$

So output will be proportional to productivity to the $\sigma = 1/(1-\rho)$ power

Cutoff Good: Case 2 Autarky

Note for the cutoff good $\pi(\bar{z}) = 0$, therefore we should have

$$0 = p(\bar{z})y(\bar{z}) - \frac{y(\bar{z})}{\bar{z}} - f$$

$$0 = \frac{1}{\rho\bar{z}}\left(\frac{\rho\big((1-\rho)\gamma - \rho\big)\theta(L+\pi)}{\mu\gamma(1-\rho)}\bar{z}^{\frac{(1-\rho)(\gamma+1)}{1-\rho}}\right) - \frac{\frac{\rho\big((1-\rho)\gamma-\rho\big)\theta(L+\pi)}{\mu\gamma(1-\rho)}\bar{z}^{\frac{(1-\rho)(\gamma+1)}{1-\rho}}}{\bar{z}} - f$$

Which simplifies to

$$\left(\frac{\big((1-\rho)\gamma - \rho\big)\theta(L+\pi)}{\mu\gamma}\bar{z}^{\gamma}\right) = f$$

And therefore we can solve for $\bar{z}$ in terms of π:

$$\bar{z} = \left(\frac{f\mu\gamma}{\big((1-\rho)\gamma-\rho\big)\theta(L+\pi)}\right)^{\frac{1}{\gamma}}$$

Total Profits: Case 2 Autarky

The last thing to solve for is total profits, note

$$\pi = \mu \int_{\bar{z}}^{\infty} \left(p(z)c(z) - \frac{c(z)}{\bar{z}} - f \right) dF(z)$$

Plugging in $p(z), c(z)$, and $dF(z)$ and simplifying yields

$$\pi = \mu \frac{\big((1-\rho)\gamma - \rho\big)\theta(L+\pi)}{\bar{z}^{\frac{\rho-(1-\rho)\gamma}{1-\rho}}} \int_{\bar{z}}^{\infty} z^{\frac{\rho}{1-\rho}-\gamma-1} dz - \mu\gamma f \int_{\bar{z}}^{\infty} z^{-\gamma-1} dz$$

$$\pi = \mu \frac{\big((1-\rho)\gamma - \rho\big)\theta(L+\pi)}{\bar{z}^{\frac{\rho-(1-\rho)\gamma}{1-\rho}}} \left(\frac{1-\rho}{\rho - \gamma(1-\rho)} \right) \left[z^{\frac{\rho}{1-\rho}-\gamma} \right]_{\bar{z}}^{\infty} - \mu\gamma f \left(-\frac{1}{\gamma} \right) \left[z^{-\gamma} \right]_{\bar{z}}^{\infty}$$

$$\pi = \mu \frac{(1-\rho)\theta(L+\pi)}{\bar{z}^{\frac{\rho-(1-\rho)\gamma}{1-\rho}}} \bar{z}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} - \mu \bar{z}^{-\gamma} f$$

Total Profits: Case 2 Autarky

Continuing to simplify yields $\pi = (1-\rho)\theta(L+\pi) - \mu\bar{z}^{-\gamma} f$; which allows us to solve for π in terms of $\bar{z}$

$$\pi = \frac{(1-\rho)\theta L - \mu\bar{z}^{-\gamma} f}{1-(1-\rho)\theta}$$

Plugging in $\bar{z}$ (which is in terms of π) yields

$$\pi = \frac{(1-\rho)\theta L - \mu\left(\frac{\left((1-\rho)\gamma-\rho\right)\theta(L+\pi)}{f\mu\gamma}\right) f}{1-(1-\rho)\theta}$$

Which we can solve for π to get

$$\pi = \frac{\rho\theta L}{\gamma-\rho\theta} \Rightarrow wL + \pi = \frac{\gamma L}{\gamma-\rho\theta}$$

Cutoff: Case 2 Autarky

Plugging profits into $\bar{z}$ and simplifying yields the cutoff productivity

$$\bar{z} = \left(\frac{f\mu(\gamma - \rho\theta)}{\big((1-\rho)\gamma - \rho\big)\theta L}\right)^{\frac{1}{\gamma}}$$

Note that if $\bar{z} < 1$ then we are not actually in case 2.

We can also find the mass of the set of firms that produce, M, (μ is the set of potential producers):

$$M = \mu\left(\int_{\bar{z}}^{\infty} dF(z)\right) = \mu \int_{\bar{z}}^{\infty} \gamma z^{-\gamma-1} dz = \mu \frac{\gamma z^{-\gamma-1+1}}{-\gamma - 1 + 1}\Bigg|_{\bar{z}}^{\infty} = \mu \bar{z}^{-\gamma}$$

Price Index: Case 1

In this case fixed costs are low enough that all firms produce.

Recall our price index from case 2 is:

$$\left(P_j\right)^{-\left(\frac{\rho}{1-\rho}\right)} = \gamma \mu_j \left(\frac{w_j}{\rho}\right)^{-\left(\frac{\rho}{1-\rho}\right)} \left(\frac{1-\rho}{(1-\rho)\gamma-\rho}\right) \bar{z}^{\frac{\rho-(1-\rho)\gamma}{1-\rho}}$$

To get the price index in case 1, we can simply substitute 1 for $\bar{z}$ in the above, therefore

$$\left(P_j\right)^{-\left(\frac{\rho}{1-\rho}\right)} = \gamma \mu_j \left(\frac{w_j}{\rho}\right)^{-\left(\frac{\rho}{1-\rho}\right)} \left(\frac{1-\rho}{(1-\rho)\gamma-\rho}\right)$$

Output and Profits: Case 1 Autarky

Can substitute prices and price index into demand to get output for firm with productivity z. Again dropping the subscript j and plugging in $w = 1$ and then simplifying gives

$$c(z) = \frac{\theta(L+\pi)}{p(z)^{\frac{1}{1-\rho}} P^{\frac{-\rho}{1-\rho}}} = \frac{\rho\big((1-\rho)\gamma - \rho\big)\theta(L+\pi)}{\mu\gamma(1-\rho)} z^{\frac{1}{1-\rho}}$$

And total profits will be again given by $\pi = \mu \int_1^\infty (p(z)c(z) - c(z) - f)dF(z)$

Which adding in prices and consumption, then integrating yields

$$\pi = (1-\rho)\theta(L+\pi) - \mu f$$

And therefore profits are equal to

$$\pi = \frac{(1-\rho)\theta L - \mu f}{1-(1-\rho)\theta} \Rightarrow wL + \pi = \frac{L - \mu f}{1-(1-\rho)\theta}$$

International Trade

Now consider the case with international trade, for simplicity suppose $N = 2$ countries

- Fixed mass of firms μ_j in each country, each firm produces a differentiated good
- Firm m has productivity $z(m)$. Distribution of $z(m)$ follows a Pareto distribution.
- Firms must pay a fixed cost, f_d, in terms of labor in order to produce goods for the domestic market, and another fixed cost on top of that, f_e, to export goods to foreign markets
  - Exports from country i to j are subject to a symmetric iceberg trade cost τ
  - Firms know their productivity before deciding to produce/export

Household Demand

Consumer's problem is same as before, but can be useful to write as

$$\max_{\{c_{j,0}, c_j(m)\}} (1-\theta)\log c_{j,0} + \frac{\theta}{\rho}\log\left(\int_{m\in M_j^d} \left(c_j(m)\right)^{\rho} dm + \int_{m\in M_i^e} \left(c_i^j(m)\right)^{\rho} dm\right)$$

Where M_j^d is the set of domestic producers in country j and M_j^e is the set of producers in country i that export to country j.

$$p_{j,0}c_{j,0} + \int_{m\in M_j^d} p_j(m)c_j(m) + \int_{m\in M_i^e} p_i^j(m)c_i^j(m) = w_jL_j + \pi_j$$

where $w_jL_j + \pi_j = I_j$ is the income of households in country j.

- Useful since can do market clearing separately for domestic output and exports

Household Demand

After taking FOC and solving for demand, we have again have

$$c_{j,0} = \frac{(1-\theta)\left(w_j L_j + \pi_j\right)}{p_{j,0}}$$

$$c_j(k) = \frac{\theta\left(w_j L_j + \pi_j\right)}{\left(p_j(k)\right)^{\frac{1}{1-\rho}} \left(P_j\right)^{-\left(\frac{\rho}{1-\rho}\right)}}, k \in M_j^d, \qquad c_i^j(k) = \frac{\theta\left(w_j L_j + \pi_j\right)}{\left(p_i^j(k)\right)^{\frac{1}{1-\rho}} \left(P_j\right)^{-\left(\frac{\rho}{1-\rho}\right)}}, k \in M_i^e$$

where P_j is the CES price index

$$P_j = \left(\int_{m \in M_j^d} \left(p_j(m)\right)^{-\left(\frac{\rho}{1-\rho}\right)} dm + \int_{m \in M_i^e} \left(p_i^j(m)\right)^{-\left(\frac{\rho}{1-\rho}\right)} dm\right)^{-\left(\frac{1-\rho}{\rho}\right)}$$

Household Demand

After taking FOC and solving for demand, we have again have

$$c_{j,0} = \frac{(1-\theta)\left(w_j L_j + \pi_j\right)}{p_{j,0}}$$

$$c_j(k) = \frac{\theta\left(w_j L_j + \pi_j\right)}{\left(p_j(k)\right)^{\frac{1}{1-\rho}} \left(P_j\right)^{-\left(\frac{\rho}{1-\rho}\right)}}, k \in M_j, \qquad c_i^j(k) = \frac{\theta\left(w_j L_j + \pi_j\right)}{\left(p_i^j(k)\right)^{\frac{1}{1-\rho}} \left(P_j\right)^{-\left(\frac{\rho}{1-\rho}\right)}}, k \in M_i^e$$

where P_j is the CES price index

$$P_j = \left(\int_{m \in M_j^d} \left(p_j(m)\right)^{-\left(\frac{\rho}{1-\rho}\right)} dm + \int_{m \in M_i^e} \left(p_i^j(m)\right)^{-\left(\frac{\rho}{1-\rho}\right)} dm\right)^{-\left(\frac{1-\rho}{\rho}\right)}$$

Firm Prices

Production function for domestic output is:

$$y_j(m) = \max\{z_j(m)\big(l_j(m) - f_d\big), 0\}$$

Conditional on $y_j(m) > 0$, production function for exported output is:

$$y_j^i(m) = \max\left\{\frac{z_j(m)}{\tau}\big(l_j^i(m) - f_e\big), 0\right\}$$

Profit maximization will again yield that prices are a constant markup over marginal cost

$$p_j(m) = \frac{w_j}{\rho z_j(m)} \ if \ y_j(m) > 0, \qquad p_j^i(z) = \tau \frac{w_j}{\rho z_j(m)} if \ y_j^i(z) > 0$$

Productivity, Fixed Costs, and Cutoffs: Trade

Suppose again there is a fixed mass of firms μ_j, and their productivity follows a Pareto distribution with cutoff 1 and tail parameter $\gamma > \rho/(1-\rho) > 2$.

Assume countries are symmetric. Three possible equilibrium cases:

Case 1: All firms produce and export and earn positive profits doing both regardless of productivity

- $y_j(m) > 0, y_j^i(m) > 0 \ \forall m$

Case 2: All firms produce domestically, cutoff productivity $\bar{z}_e$ for exporting

- $y_j(m) > 0 \ \forall m, \ y_j^i(m) > 0$ iff $z_j(m) \geq \bar{z}_e$

Case 3: Cutoff productivity $\bar{z}_d$ for producing domestically and cutoff productivity $\bar{z}_e$ for exporting

- $y_j(m) > 0$ iff $z_j(m) \geq \bar{z}_d, \ y_j^i(m) > 0$ iff $z(m) \geq \bar{z}_e; \ \bar{z}_e > \bar{z}_d$

Price Index with Pareto Distribution for Productivity: Case 3

Assume we are in case 3, so there are cutoffs $\bar{z}_d$ and $\bar{z}_e$ for producing domestically and exporting

Label firms by productivity again, and rewrite price index equation as

$$P_j^{\frac{-\rho}{1-\rho}} = \mu_j \int_{\bar{z}_d}^{\infty} p_j(z)^{\frac{-\rho}{1-\rho}} dF(z) + \mu_i \int_{\bar{z}_e}^{\infty} p_i^j(z)^{\frac{-\rho}{1-\rho}} dF(z)$$

Computing the integrals and solving yields

$$P_j^{\frac{-\rho}{1-\rho}} = \gamma \rho^{\frac{\rho}{1-\rho}} \left(\frac{1-\rho}{(1-\rho)\gamma - \rho} \right) \left[\mu_j \bar{z}_d^{\frac{\rho-(1-\rho)\gamma}{1-\rho}} + \mu_i \tau^{\frac{-\rho}{1-\rho}} \bar{z}_e^{\frac{\rho-(1-\rho)\gamma}{1-\rho}} \right]$$

Cutoffs and Profits: Case 3

Domestic demand for firm with productivity z will be

$$c_j(z) = \frac{\rho\big((1-\rho)\gamma-\rho\big)\theta\big(L+\pi_j\big)}{\gamma(1-\rho)\left[\mu_j \bar{z}_d^{\frac{\rho-(1-\rho)\gamma}{1-\rho}} + \mu_i \tau^{\frac{-\rho}{1-\rho}} \bar{z}_e^{\frac{\rho-(1-\rho)\gamma}{1-\rho}}\right]} z^{\frac{1}{1-\rho}}$$

The cutoff productivity for domestic profits $\bar{z}_d$ satisfies zero profits from domestic production

$$0 = p_j(\bar{z}_d)c_j(\bar{z}_d) - \frac{c_j(\bar{z}_d)}{\bar{z}_d} - f_d$$

Plugging in prices and demand yields

$$\frac{\big((1-\rho)\gamma-\rho\big)\theta\big(L+\pi_j\big)}{\gamma\left[\mu_j \bar{z}_d^{\frac{\rho-(1-\rho)\gamma}{1-\rho}} + \mu_i \tau^{\frac{-\rho}{1-\rho}} \bar{z}_e^{\frac{\rho-(1-\rho)\gamma}{1-\rho}}\right]} \bar{z}_d^{\frac{\rho}{1-\rho}} - f_d = 0$$

Cutoffs and Profits: Case 3

Foreign demand for firm with productivity z will be

$$c_i^j(z) = \frac{\rho\big((1-\rho)\gamma - \rho\big)\theta\big(L+\pi_j\big)}{\gamma\tau^{\frac{1}{1-\rho}}(1-\rho)\left[\mu_j \overline{z}_d^{\frac{\rho-(1-\rho)\gamma}{1-\rho}} + \mu_i \tau^{\frac{-\rho}{1-\rho}} \overline{z}_e^{\frac{\rho-(1-\rho)\gamma}{1-\rho}}\right]} z^{\frac{1}{1-\rho}}$$

The cutoff productivity for foreign profits $\bar{z}_e$ satisfies zero profits from exporting

$$0 = p_j^i(\overline{z}_e)c_j^i(\overline{z}_e) - \frac{c_j^i(\overline{z}_e)}{\overline{z}_e} - f_e$$

Plugging in prices and demand yields

$$\frac{\big((1-\rho)\gamma - \rho\big)\theta\big(L+\pi_j\big)}{\gamma\tau^{\frac{1}{1-\rho}}\left[\mu_j \overline{z}_d^{\frac{\rho-(1-\rho)\gamma}{1-\rho}} + \mu_i \tau^{\frac{-\rho}{1-\rho}} \overline{z}_e^{\frac{\rho-(1-\rho)\gamma}{1-\rho}}\right]} \overline{z}_e^{\frac{\rho}{1-\rho}} - f_e = 0$$

Total Profits: Case 3

Total profits is sum of profits from domestic output and profits from exporting for each firm

$$\pi_j = \mu_j \int_{\bar{z}_d}^{\infty} \left(p_j(z) c_j(z) - \frac{c_j(z)}{z} - f_d \right) dF(z) + \mu_j \int_{\bar{z}_e}^{\infty} \left(p_j^i(z) c_j^i(z) - \frac{c_j^i(z)}{z} - f_e \right) dF(z)$$

Plugging in $dF(z) = (\gamma z^{-\gamma-1})dz$ and imposing symmetry so that $\mu_j = \mu, L_j = L$ we get (drop j)

$$\pi = \mu \int_{\bar{x}_d}^{\infty} \left(\frac{\big((1-\rho)\gamma - \rho\big)\theta(L+\pi)}{\gamma \left[\mu \bar{z}_d^{\frac{\rho-(1-\rho)\gamma}{1-\rho}} + \mu \tau^{\frac{-\rho}{1-\rho}} \bar{z}_e^{\frac{\rho-(1-\rho)\gamma}{1-\rho}} \right]} z^{\frac{\rho}{1-\rho}} - f_d \right) (\gamma z^{-\gamma-1}) dz + \mu \int_{\bar{x}_e}^{\infty} \left(\frac{\big((1-\rho)\gamma - \rho\big)\theta(L+\pi)}{\gamma \tau^{\frac{\rho}{1-\rho}} \left[\mu \bar{z}_d^{\frac{\rho-(1-\rho)\gamma}{1-\rho}} + \mu \tau^{\frac{-\rho}{1-\rho}} \bar{z}_e^{\frac{\rho-(1-\rho)\gamma}{1-\rho}} \right]} z^{\frac{\rho}{1-\rho}} - f_e \right) (\gamma z^{-\gamma-1}) dz$$

Computing the integral and simplifying the previous expression for π yields

$$\pi = (1-\rho)\theta(L+\pi) - \mu\left(\bar{z}_d{}^{-\gamma} f_d + \bar{z}_e{}^{-\gamma} f_e\right)$$

Relative Cutoffs: Case 3

From our zero profit conditions we have cutoffs we have, $\bar{z}_d$ satisfies

$$\frac{\big((1-\rho)\gamma-\rho\big)\theta(L+\pi)}{\gamma\left[\mu\bar{z}_d^{\frac{\rho-(1-\rho)\gamma}{1-\rho}}+\mu\tau^{\frac{-\rho}{1-\rho}}\bar{z}_e^{\frac{\rho-(1-\rho)\gamma}{1-\rho}}\right]}\bar{z}_d^{\frac{\rho}{1-\rho}}-f_d=0$$

While $\bar{z}_e$ satisfies

$$\frac{\big((1-\rho)\gamma-\rho\big)\theta(L+\pi)}{\gamma\tau^{\frac{\rho}{1-\rho}}\left[\mu\bar{z}_d^{\frac{\rho-(1-\rho)\gamma}{1-\rho}}+\mu\tau^{\frac{-\rho}{1-\rho}}\bar{z}_e^{\frac{\rho-(1-\rho)\gamma}{1-\rho}}\right]}\bar{z}_e^{\frac{\rho}{1-\rho}}-f_e=0$$

Therefore relative cutoffs are

$$\frac{\bar{z}_d^{\frac{\rho}{1-\rho}}}{\bar{z}_e^{\frac{\rho}{1-\rho}}}=\frac{f_d}{f_e}\frac{1}{\tau^{\frac{\rho}{1-\rho}}}\;\Rightarrow\;\frac{\bar{z}_d}{\bar{z}_e}=\frac{1}{\tau}\left(\frac{f_d}{f_e}\right)^{\frac{1-\rho}{\rho}}$$

Domestic Cutoff: Case 3

Plugging in the relative cutoffs means that $\bar{z}_d$ satisfies

$$\frac{\big((1-\rho)\gamma-\rho\big)\theta(L+\pi)}{\gamma\left[\mu\bar{z}_d^{\frac{\rho-(1-\rho)\gamma}{1-\rho}}+\mu\tau^{\frac{-\rho}{1-\rho}}\left(\tau\left(\frac{f_e}{f_d}\right)^{\frac{1-\rho}{\rho}}\bar{z}_d\right)^{\frac{\rho-(1-\rho)\gamma}{1-\rho}}\right]}\bar{z}_d^{\frac{\rho}{1-\rho}}-f_d=0$$

Simplifying yields

$$\bar{z}_d^{\gamma}=f_d\left(\frac{\mu\gamma\left[1+\tau^{-\gamma}\left(\frac{f_e}{f_d}\right)^{\frac{\rho-(1-\rho)\gamma}{\rho}}\right]}{\big((1-\rho)\gamma-\rho\big)\theta(L+\pi)}\right)$$

Which we can then use to get $\bar{z}_e$ in terms of π

Profits: Case 3

Plugging in the cutoffs into profits yields

$$\pi = (1-\rho)\theta(L+\pi) - \mu\left(\left(\frac{\big((1-\rho)\gamma-\rho\big)\theta(L+\pi)}{\mu\gamma\left[1+\tau^{-\gamma}\left(\frac{f_e}{f_d}\right)^{\frac{\rho-(1-\rho)\gamma}{\rho}}\right]f_d}\right)f_d\left(1+\frac{1}{f_d}\tau^{-\gamma}\left(\frac{f_d}{f_e}\right)^{\gamma\frac{1-\rho}{\rho}}f_e\right)\right)$$

Which simplifies into

$$\pi = \theta(L+\pi)\left(\frac{\rho}{\gamma}\right)$$

And therefore, we end up with the same profit as in autarky case 2

$$\pi = \frac{\theta L\rho}{\gamma-\theta\rho}$$

Cutoffs: Case 3

We can then plug profits into our cutoffs to get

$$\bar{z}_d^{\gamma} = f_d \left(\frac{\mu\gamma \left[1 + \tau^{-\gamma} \left(\frac{f_e}{f_d} \right)^{\frac{\rho-(1-\rho)\gamma}{\rho}} \right]}{((1-\rho)\gamma - \rho)\theta \left(\frac{\gamma \bar{l}}{\gamma - \theta\rho} \right)} \right)$$

And

$$\bar{z}_e^{\gamma} = \left(\tau \left(\frac{f_e}{f_d} \right)^{\frac{1-\rho}{\rho}} \right)^{\gamma} f_d \left(\frac{\mu\gamma \left[1 + \tau^{-\gamma} \left(\frac{f_e}{f_d} \right)^{\frac{\rho-(1-\rho)\gamma}{\rho}} \right]}{((1-\rho)\gamma - \rho)\theta \left(\frac{\gamma \bar{l}}{\gamma - \theta\rho} \right)} \right)$$

Now we have all variables in terms of model parameters