ECON 256: Poverty, Growth & Inequality

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Much Ado About Formulas

We live in a world with computers and the internet. What does this mean?

- Memorizing tons of formulas is a waste of time
- Thoughtlessly plugging formulas into a calculator isn’t much better

What isn’t a waste of time? (What don’t computers do better than you?)

- Understanding what the formulas represent and how they should be used
- Knowing how the formulas were derived, and how to derive new formulas

Deriving Growth Rate Formulas

Where did the growth rate formulas come from?

There are only two things you really need to have memorized/internalized.

Formula 1: If something grows by 5 percent per year for three years then

$$\text{Value in 3 years} = (1.05) \times (1.05) \times (1.05) \times (\text{Value now})$$

- Have to understand what 5 **percent** growth means and that growth rates are **multiplicative**.

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Formula 2: If we have Value before and Value after then the total growth in percent is

$$\text{Total growth (\%)} = 100 \times \left(\frac{V_{\text{after}} - V_{\text{before}}}{V_{\text{before}}}\right) = 100 \times \left(\left(\frac{V_{\text{after}}}{V_{\text{before}}}\right) - 1\right)$$

- Have to understand that the 100 transforms it from ratio to % (also called **percent change**), and that the initial value is the denominator, and that the two ways of writing it are equivalent.

Deriving Growth Rate Formulas

If you understand those, you can get all the other formulas with simple algebra.

First, let's generalize the first formula.

- If something grows by 5 percent per year for three years then

$$\text{Value in 3 years} = (1.05) \times (1.05) \times (1.05) \times (\text{Value now})$$

- **Updated Formula 1:** This means, if something grows by r percent per year over N years then

$$\text{Value in N years} = \left(\frac{100 + r}{100}\right)^{N} \times (\text{Value now})$$

Deriving Growth Rate Formulas

Now we can do things like combine Formula 1 and Formula 1

Formula 1: If something grows by r percent per year over N years then

$$\text{Value in N years} = \left(\frac{100 + r}{100}\right)^N \times (\text{Value now})$$

That means (divide both sides by Value now)

$$\frac{\text{Value in N years}}{\text{Value now}} = \left(\frac{100 + r}{100}\right)^N$$

We can plug the above into Formula 2 (value after = Value in N years; value before = Value now)

Deriving Growth Rate Formulas

We can plug the previous blue thing directly into Thing 2 and we get

New Formula: The total growth after N years, in percent, is given by (this is formula on slide 23)

$$\text{Total growth after N years} = 100 \times \left(\left(\frac{\text{Value in N years}}{\text{Value now}} \right) - 1 \right) = 100 \times \left(\left(\frac{100 + r}{100} \right)^{N} - 1 \right)$$

Deriving Growth Rate Formulas

We can also rearrange formulas, depending what we're interested in finding

From **Formula 1:** If something grows by r percent per year over N years then

$$\frac{\text{Value in N years}}{\text{Value now}} = \left(\frac{100 + r}{100}\right)^N$$

Taking both sides to the $1/N$ power

$$\left(\frac{\text{Value in N years}}{\text{Value now}}\right)^{\frac{1}{N}} = \frac{100 + r}{100}$$

Multiplying both sides by 100

$$100 \times \left(\frac{\text{Value in N years}}{\text{Value now}}\right)^{\frac{1}{N}} = 100 + r$$

Deriving Growth Rate Formulas

$$100 \times \left(\frac{\text{Value in N years}}{\text{Value now}}\right)^{\frac{1}{N}} = 100 + r$$

Subtracting 100 from both sides

$$100 \times \left(\frac{\text{Value in N years}}{\text{Value now}}\right)^{\frac{1}{N}} - 100 = r$$

Combining terms and flipping the LHS and RHS gives us a formula for r

New Formula 2: Assuming exponential growth, we can find the annualized growth rate, r, from

$$r = 100 \times \left(\left(\frac{\text{Value in N years}}{\text{Value now}}\right)^{\frac{1}{N}} - 1\right)$$

(Note this is exact same equation as Formula 1, just rearranged. This is formula on slide 26)

Using Formulas to Answer New Questions

Convergence is a central question in development

- Should we expect poor countries catch up to rich countries? How long should it take?
- How does actual progress compare with expected progress.

I didn’t have a formula in the slides (prior to Friday) to answer questions on convergence

- It’s also not in the textbook and it’s not easy to google either.
- What do we do?????!?!?!?!?!?!?

Using Formulas to Answer New Questions

Example Question

- Suppose a rich country grows at 2% per year and is 10 times richer than a poor country. If the poor country grows at 5% per year, approximately how many years will it take for it to catch up to the rich country?

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Example Question

- Suppose a rich country grows at 2% per year and is 10 times richer than a poor country. If the poor country grows at 5% per year, approximately how many years will it take for it to catch up to the rich country?

- Everything we need to know is in Formula 1.

$$\text{Value in N years} = \left(\frac{100 + r}{100}\right)^{N} \times (\text{Value now})$$

- Let $V_{P,N}$ be the GDP per capita for the poor country after N years, then formula 1 says

$$V_{P,N} = \left(\frac{100 + 5}{100}\right)^{N} V_{P,0}$$

Calculating Time to Convergence

- Formula 1 for poor country

$$V_{P,N} = \left(\frac{100 + 5}{100}\right)^N V_{P,0}$$

- Likewise, let $V_{R,N}$ be GDP per capita of rich country after N years. Formula 1 gives

$$V_{R,N} = \left(\frac{100 + 2}{100}\right)^N V_{R,0}$$

- Now the question is: what is N such that $V_{P,N} = V_{R,N}$?

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- Now the question is: what is N such that $V_{P,N} = V_{R,N}$?

- If $V_{P,N} = V_{R,N}$ then we must have the RHS of both equations equal each other too:

$$\left(\frac{100+5}{100}\right)^N V_{P,0} = \left(\frac{100+2}{100}\right)^N V_{R,0}$$

Calculating Time to Convergence

$$\left(\frac{100+5}{100}\right)^N V_{P,0} = \left(\frac{100+2}{100}\right)^N V_{R,0}$$

Dividing both sides by $V_{P,0}$ gives

$$\left(\frac{100+5}{100}\right)^N = \left(\frac{100+2}{100}\right)^N \frac{V_{R,0}}{V_{P,0}}$$

And remember, the rich country is currently 10 times richer, so $V_{R,0}/V_{P,0} = 10$, so we have

$$\left(\frac{100+5}{100}\right)^N = \left(\frac{100+2}{100}\right)^N \times 10$$

Calculating Time to Convergence

$$\left(\frac{100+5}{100}\right)^N = \left(\frac{100+2}{100}\right)^N \times 10$$

Now lets take logs of both sides to bring the exponents down

$$\log\left[\left(\frac{100+5}{100}\right)^N\right] = \log\left[\left(\frac{100+2}{100}\right)^N \times 10\right]$$

Calculating Time to Convergence

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Now lets take logs of both sides to bring the exponents down

$$\log\left[\left(\frac{100+5}{100}\right)^N\right] = \log\left[\left(\frac{100+2}{100}\right)^N \times 10\right]$$

Using the properties of logs we have (recall $\log(XY) = \log X + \log Y$ and $\log X^a = a \log X$)

$$\log\left[\left(\frac{100+5}{100}\right)^N\right] = \log\left[\left(\frac{100+2}{100}\right)^N\right] + \log[10]$$

$$N \log\left[\left(\frac{100+5}{100}\right)\right] = N \log\left[\left(\frac{100+2}{100}\right)\right] + \log[10]$$

Calculating Time to Convergence

$$N \log\left[\left(\frac{100+5}{100}\right)\right] = N \log\left[\left(\frac{100+2}{100}\right)\right] + \log[10]$$

Subtracting $N \log\left[\left(\frac{100+2}{100}\right)\right]$ from both sides gives

$$N \log\left[\left(\frac{100+5}{100}\right)\right] - N \log\left[\left(\frac{100+2}{100}\right)\right] = \log[10]$$

Calculating Time to Convergence

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Factoring out the N gives

$$N\left(\log\left[\left(\frac{100+5}{100}\right)\right] - \log\left[\left(\frac{100+2}{100}\right)\right]\right) = \log[10]$$

Dividing both sides by expression inside the parenthesis gives

$$N = \log[10] \,/\, \left(\log\left[\left(\frac{100+5}{100}\right)\right] - \log\left[\left(\frac{100+2}{100}\right)\right]\right)$$

Calculating Time to Convergence

Now we have something we can plug into our calculator (or better, Excel)

$$N = \log[10] \,/\, \left(\log\left[\left(\frac{100+5}{100}\right)\right] - \log\left[\left(\frac{100+2}{100}\right)\right] \right)$$

(Since calculators typically do log base 10, that's what I'll use. Often we prefer to use the natural log, ln or log base e, in economics. Either one will give the correct answer here)

Calculating Time to Convergence

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$$N = \frac{1}{(0.0212 - 0.0086)}$$

$$= \frac{1}{.0126} \approx 79.4$$

Which is very close to the given answer of 80. (80 years until the first year where the poor country is richer at the start of the year)

General Formula for Time to Convergence

We can use the same steps to find the general formula:

$$\text{Time to Convergence } (N) = \frac{\log\left[\frac{V_{R,0}}{V_{P,0}}\right]}{\left(\log\left[\left(\frac{100 + r_P}{100}\right)\right] - \log\left[\left(\frac{100 + r_R}{100}\right)\right]\right)}$$

Where $\frac{V_{R,0}}{V_{P,0}}$ is how much richer the rich country is initially, r_P is the growth rate of the poor country, and r_R is the growth rate of the rich country.

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Where $\frac{V_{R,0}}{V_{P,0}}$ is how much richer the rich country is initially, r_P is the growth rate of the poor country, and r_R is the growth rate of the rich country.

- We can simplify slightly more to get the formula in the “time to convergence” notes

$$\text{Time to Convergence } (N) = \frac{\log\left[\frac{V_{R,0}}{V_{P,0}}\right]}{\log\left[\left(\frac{100 + r_P}{100 + r_R}\right)\right]}$$

Why Bother?

Now that we have the formula, why do we need to know how to derive it?

1. Nobody is going to remember that formula off the top of their head
2. It’s not an easy formula to look up. Referring to my website for the rest of your life isn’t practical.
3. You’re stuck if the question is slightly different and the formula doesn’t apply, but not if you know how to derive the formula you need
4. You understand things a lot better when you know where they came from.

Understanding things is your comparative advantage

IMPORTANT

- You will never get good at manipulating formulas unless you do it on your own
- Looking over my steps and thinking “makes sense” does not mean you know it
- Try to derive it yourself. Check my notes only if you get stuck
- Trying to re-deriving things yourself is a great way to get better

Economists actually do need to manipulate formulas in the real world

- For you, Math = $$$
- Also, learn computers