ECO 745 Notes: Alwyn Young (1991) Autarky Equilibrium Derivations

Economy Setup

•There are two countries: $i, j = 1,2$

•A continuum of goods: $z \in [0, \infty]$

•Labor is the only factor of production, and is supplied inelastically.

•Non-homothetic preferences over goods. Not all goods consumed.

•No trade costs (price of goods will be same in both countries)

•Dynamic model with learning-by-doing spillovers

•No saving

Learning-by-doing Equations

Still have same general production technology

$$y_i(z) = \frac{l_i(z)}{a_i(z)}$$

Except we now impose a minimal bound on unit labor requirements for each good

$$a_i(z,t) \geq \bar{a}(z) = e^{-z}$$

We have the following learning-by-doing equation

$$\frac{\dot{a}_i(z,t)}{a_i(z,t)} = \begin{cases} -\int_0^\infty b_i(z',t) l_i(z',t) dz' \ , & if\ a_i(z,t) > \bar{a}(z) \\ 0, & if\ a_i(z,t) > \bar{a}(z) \end{cases}$$

Where

$$b_i(z',t) = \begin{cases} b, & if\ a_i(z',t) > \bar{a}(z') \\ 0, & if\ a_i(z',t) = \bar{a}(z') \end{cases}$$

Here $\dot{a}_i(z,t) \coloneqq \partial a_i(z,t)/\partial t$, and ($b > 0$).

Initial State:

At $t_0 = 0$ let there be $z_{i,\,0}$ such that

$$a_i(z,0) = \begin{cases} \bar{a}(z), & if\ z \leq z_{i,0} \\ e^{z-2z_{i,0}}, & if\ z > z_{i,0} \end{cases}$$

Where again $\bar{a}(z) = e^{-z}$

Autarky Equilibrium

1. Consumers Problem:

Maximize discounted utility (the discount rate is only important for computing welfare):

$$\max \int_0^\infty e^{-\rho t} \left[\int_0^\infty \log(c_i(z,t)+1)dz \right] dt$$

Subject to budget constraint in each period

$$\int_0^\infty p_i(z,t)c_i(z,t)dz = w_i(t)L_i(t)\,, \ \ \forall t$$

2. Firms Problem:

Maximize profits in each period t:

$$\max p_i(z,t)y_i(z,t) - w_i(t)l_i(z,t)$$

Subject to technology

$$y_i(z,t) = \frac{l_i(z,t)}{a_i(z,t)}$$

3. Market Clearing

Goods market clears in each period (here we're in autarky)

$$c_i(z,t) = y_i(z,t), \ \ \forall t$$

Labor market clears

$$\int_0^\infty l_i(z,t)dz = L_i(t)$$

4) Learning-by-doing equations are satisfied

Solving the Autarky Equilibrium (drop country subscripts)

So at any time period t, the FOC from the consumer's problem gives

$$\frac{1}{c(x,t)+1} - p(x,t)\lambda(t) \leq 0, \qquad w.e.if\ c(x,t) > 0$$

Therefore for any two goods x and y, we have relative consumption satisfies

$$\frac{c(y,t)+1}{c(x,t)+1} = \frac{p(x,t)\lambda(t)}{p(y,t)\lambda(t)}$$

and plugging in prices from the firms problem $\big(p(z,t) = w(t)a(z,t)\big)$ gives

$$\frac{c(y,t)+1}{c(x,t)+1} = \frac{a(x,t)w(t)}{a(y,t)w(t)}$$

Therefore

$$[c(x,t)+1]a(x,t) = [c(y,t)+1]a(y,t)$$

Cutoff Goods

We know that there will be a good $M(t)$, such that for all $a(x,t) \geq a(M(t),t) = \overline{a}(M(t))$ we have $c(x,t) = 0$, and for $a(x,t) < \overline{a}(M(t))$ we have $c(x,t) > 0$. Therefore that implies that

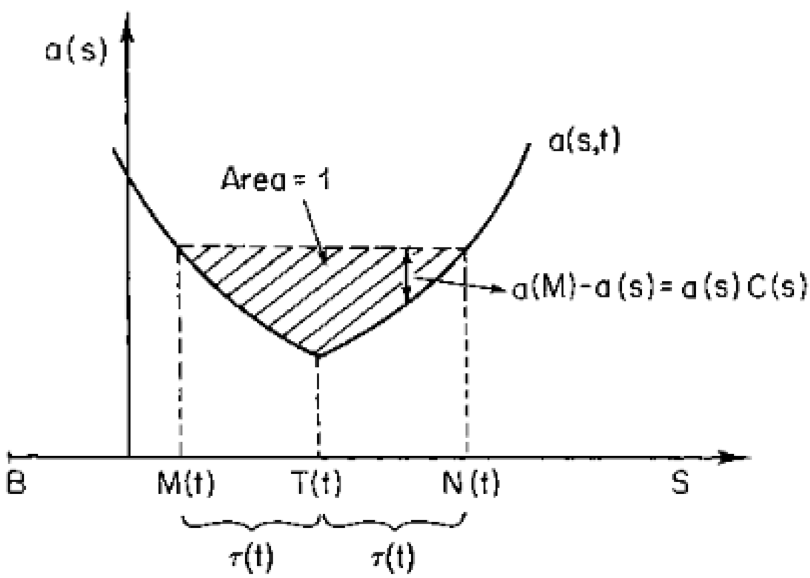
$$[c(x,t)+1]a(x,t) = [c(M(t),t)+1]\overline{a}(M(t))$$

$$[c(x,t)+1]a(x,t) = \overline{a}(M(t))$$

$$a(x,t)c(x,t) = \overline{a}(M(t)) - a(x,t)$$

And since $\overline{a}(x)$ is a symmetric reflection of $a(x,t)$ around $T(t)$, therefore we have a corresponding $N(t)$ with $a(N(t),t) = \overline{a}(M(t))$ so therefore the consumer consumes all goods in the interval $[M(t), N(t)]$ at point t.

Autarky Picture (from Young (1991)):



Range of goods consumed

For the budget constraint that means the consumer consumes (normalize $w_i(t) = 1$):

$$L(t) = \int_{M(t)}^{N(t)} p(x,t)c(x,t)dx = \int_{M(t)}^{N(t)} a(x,t)c(x,t)dx = \int_{M(t)}^{N(t)} [\overline{a}(M(t)) - a(x,t)]L(t)dx$$

Then dividing both sides by $L_i(t)$ gives

$$1 = \int_{M(t)}^{T(t)} \left[\overline{a}\big(M(t)\big) - a(x,t)\right]dx + \int_{T(t)}^{N(t)} [a(N(t),t) - a(x,t)]dx$$

Bringing the parts that don't depend on x out of the integrals

$$= \overline{a}\big(M(t)\big)[T(t) - M(t)] + a(N(t),t)[N(t) - T(t)] - \int_{M(t)}^{T(t)} [a(x,t)]dx - \int_{T(t)}^{N(t)} [a(x,t)]dx$$

Plugging in the labor productivities

$$= e^{-M(t)}[T(t) - M(t)] + e^{N(t)-2T(t)}[N(t) - T(t)] - \int_{M(t)}^{T(t)} [e^{-x}]dx - \int_{z(t)}^{N(t)} \left[e^{x-2T(t)}\right]dx$$

$$= e^{-M(t)}[T(t) - M(t)] + e^{N(t)-2T(t)}[N(t) - z(t)] - \left[-e^{-T(t)} + e^{-M(t)}\right] - \left[e^{N(t)-2T(t)} - e^{-T(t)}\right]$$

and using $T(t) - M(t) = N(t) - T(t)$ and $e^{-M(t)} = e^{N(t)-2T(t)}$:

$$= e^{-M(t)}[T(t) - M(t)] + e^{-M(t)}[T(t) - M(t)] + e^{-T(t)} - e^{-M(t)} - e^{-M(t)} + e^{-T(t)}$$

$$= 2e^{-M(t)}[T(t) - M(t)] - 2e^{-M(t)} + 2e^{-T(t)}$$

therefore if we set $\tau(t) = T(t) - M(t)$ as the "radius" of the range of goods consumed then

$$e^{T(t)} = 2(\tau(t) - 1)e^{\tau(t)} + 2$$

so as $z(t)$ increases the range of goods consumed increases and shifts to the right

Growth rate of $T(t)$

First let's examine the growth rate of the $T(t)$, we have from the learning by doing equations

$$\frac{\dot{a}_i(T(t),t)}{a_i(T(t),t)} = \frac{de^{-T(t)}/dt}{e^{T(t)}} = -\frac{dT(t)}{dt}$$

Therefore

$$\frac{dT(t)}{dt} = \int_{T(t)}^{\infty} bl(x,t)dx = \int_{T(t)}^{N(t)} ba(x,t)c(x,t)dx$$

$$= b\int_{T(t)}^{N(t)} [a(N(t),t) - a(x,t)]\frac{L}{2}dx = \frac{bL}{2}\left[e^{N(t)-2T(t)}[N(t) - T(t)] - \left[e^{N(t)-2T(t)} - e^{-T(t)}\right]\right]$$

$$= \frac{bL}{2}\left[e^{N(t)-2z(t)}[N(t) - T(t)] - \left[e^{N(t)-2T(t)} - e^{-T(t)}\right]\right]$$

$$= \frac{bL}{2}\left[\frac{1}{2}\left(2e^{-M(t)}[T(t) - M(t)] - 2e^{-M(t)} + 2e^{-T(t)}\right)\right]$$

$$= \frac{bL}{2}\left[\frac{1}{2}(1)\right]$$

So therefore

$$\frac{dT(t)}{dt} = \frac{bL}{4}$$

Real GDP per Capita growth rate

Define the real GDP per capita growth rate as

$$g(t) := \frac{\int_0^\infty p(x,t)\dot{y}(x,t)dx}{\int_0^\infty p(x,t)y(x,t)\,dx} - \frac{\frac{dL(t)}{dt}}{L(t)}$$

Note that we have from the budget constrain that $\int_0^\infty p(x,t)y(x,t)\,dx = L(t)$ (since $w(t) = 1$), therefore

$$\frac{dL(t)}{dt} = \frac{d\left[\int_0^\infty p(x,t)y(x,t)\,dx\right]}{dt} = \int_0^\infty \dot{p}(x,t)y(x,t)\,dx + \int_0^\infty p(x,t)\dot{y}(x,t)\,dx$$

Where again $\dot{y}(x,t) = \partial y(x,t)/\partial t$. Therefore

$$g(t) = \frac{\int_0^\infty p(x,t)\dot{y}(x,t)dx}{L(t)} - \frac{\left(\int_0^\infty \dot{p}(x,t)y(x,t)\,dx + \int_0^\infty p(x,t)\dot{y}(x,t)\,dx\right)}{L(t)}$$

$$= -\frac{\int_0^\infty \dot{p}(x,t)y(x,t)\,dx}{L(t)}$$

And plugging in prices makes it

$$= -\frac{\int_0^\infty -\dot{a}(x,t)y(x,t)dx}{L(t)}$$

Rearranging and plugging in the cutoffs and market clearing

$$= \frac{-\int_{T(t)}^{N(t)} \frac{\dot{a}(x,t)}{a(x,t)} a(x,t)c(x,t)dx}{L(t)}$$

Plugging in the learning-by-doing equation:

$$= \frac{-\int_{T(t)}^{N(t)} \left(-b\int_{T(t)}^{\infty} l(v,t)dv\right) a(x,t)c(x,t)dx}{\bar{l}}$$

Exploiting symmetry around $T(t)$ in labor used and goods consumed:

$$= \frac{b\left(\frac{1}{2}L(t)\right)\left(\frac{1}{2}\int_{M(t)}^{N(t)} a(x,t)c(x,t)dx\right)}{L(t)}$$

Putting in the budget constraint and simplifying

$$= \frac{b\left(\frac{1}{2}L(t)\right)\left(\frac{1}{2}L(t)\right)}{L(t)}$$

$$= b\left(\frac{1}{4}L(t)\right)$$

Which means the growth rate of the economy is:

$$g(t) = \frac{bL(t)}{4}$$

Note that this means there are scale effects. The bigger the labor supply the faster the economy grows.