

ECO 445/545 Notes: Deriving DFS (1979) equilibrium for a symmetric world

Economy Setup

- There are two countries: $i, j = 1, 2$
- A continuum of goods: $z \in [0, 1]$
- Labor is the only factor of production, and is supplied inelastically. $L_1 = L_2 = L$
- Cobb Douglas preferences over goods
- Countries differ only in labor productivities for each good

$$y_j(z) = l_j(z)/a_j(z)$$

$$a_1(z) = e^{az}$$

$$a_2(z) = e^{a(1-z)}$$

- No trade costs (this implies price of goods will be same in both countries)

Equilibrium Definition:

An equilibrium in this economy is prices $\{\hat{p}(z)\}_{z \in [0, 1]}$, wages \hat{w}_1, \hat{w}_2 , consumption $\{\hat{c}_j(z)\}_{z \in [0, 1]}$, output $\{\hat{y}_j(z)\}_{z \in [0, 1]}$, and labor allocation $\{\hat{l}_j(z)\}_{z \in [0, 1]}$ such that

1. Consumers Problem

Given $\hat{p}(z), \hat{w}_j$ consumers in country j chooses consumption to maximize utility

$$\max \int_0^1 \log(c_j(z)) dz$$

Subject to their budget constraint and non-negative consumption

$$\int_0^1 \hat{p}(z)c_j(z) dz = \hat{w}_j L_j$$

$$c_j(z) \geq 0, \forall z$$

2. Firms Problem

Given $\hat{p}(z), \hat{w}_j$, firms in country j chooses output and labor input to maximize profits for each good z :

$$\max \hat{p}(z)y_j(z) - \hat{w}_j l_j(z)$$

Subject to the technology constraint

$$y_j(z) = \frac{l_j(z)}{a_j(z)}$$

3. Market Clearing

The goods market clears:

$$\hat{c}_1(z) + \hat{c}_2(z) = \hat{y}_1(z) + \hat{y}_2(z), \quad \forall z \in [0,1]$$

The labor market clears:

$$\int_0^1 \hat{l}_j(z) dz = L_j, \quad j = 1,2$$

Solving Equilibrium

From consumers problem we can set up the Lagrangian

$$\mathcal{L} = \int_0^1 \log c_j(z) dz + \lambda_j \left(w_j L - \int_0^1 \hat{p}(z) c_j(z) dz \right)$$

And take the FOC with respect to $c_j(z)$ ($[X]$: is personal notation for taking derivative with respect to X)

$$[c_j(z)]: \frac{1}{c_j(z)} - \lambda_j \hat{p}(z) = 0$$

Using that FOC twice for two separate goods, z and z' , then taking the ratio yields:

$$\left(\frac{1}{c_j(z)} \right) / \left(\frac{1}{c_j(z')} \right) = (\lambda_j \hat{p}(z)) / (\lambda_j \hat{p}(z'))$$

$$c_j(z) = \frac{\hat{p}(z') c_j(z')}{\hat{p}(z)}$$

We can do that for all goods $z \in [0,1]$, and then substitute those formulas in the budget constraint to get:

$$\int_0^1 \hat{p}(z) \frac{\hat{p}(z') c_j(z')}{\hat{p}(z)} dz = \hat{w}_j L_j$$

And simplifying we get

$$\hat{p}(z') c_j(z') \int_0^1 1 dz = \hat{w}_j L_j$$

And that holds for any arbitrary good z' , therefore we have $\forall z \in [0,1]$:

$$\hat{c}_j(z) = \frac{\hat{w}_j L_j}{\hat{p}(z)}, \quad (1)$$

Using the Firm's Problem

Now we can solve the firm's problem for each country. Rearranging the technology function we have

$$l_j(z) = a_j(z)y_j(z) = \begin{cases} e^{az}, & \text{if } j = 1 \\ e^{a(1-z)}, & \text{if } j = 2 \end{cases}$$

And plugging that into the firm maximization problem gives:

$$\max \hat{p}(z)y_j(z) - w_j (a_j(z)y_j(z))$$

Taking the FOC wrt $y_1(z)$ for country 1 gives:

$$[y_1(z)]: \hat{p}(z) - a_1(z)w_1 \leq 0; \quad \text{w. e. if } y_1(z) > 0$$

And taking the FOC wrt $y_2(z)$ for country 2 gives:

$$[y_2(z)]: \hat{p}(z) - a_2(z)w_2 \leq 0; \quad \text{w. e. if } y_2(z) > 0$$

And firms only produce if they can earn non-negative profits, therefore prices will be

$$\hat{p}(z) = \min\{w_1 a_1(z), w_2 a_2(z)\}$$

Or, plugging in labor productivities:

$$\hat{p}(z) = \min\{w_1 e^{az}, w_2 e^{a(1-z)}\}, \quad (2)$$

Now, we know there will be a cutoff good \bar{z} such that Z_j is the set of goods that country j produces, where:

$$Z_1 = [0, \bar{z}]$$

$$Z_2 = [\bar{z}, 1]$$

Note at z' the cost of producing the good will be the same in both countries, therefore:

$$w_1 e^{a\bar{z}} = w_2 e^{a(1-\bar{z})}$$

Which implies that relative wages must satisfy:

$$\frac{w_1}{w_2} = e^{a-2a\bar{z}}, \quad (3)$$

Using Market Clearing

Going back to market clearing conditions, we have

$$\hat{c}_1(z) + \hat{c}_2(z) = \hat{y}_1(z) + \hat{y}_2(z)$$

And every good (except possibly good z') is only produced by one country, we have:

$$\hat{c}_1(z) + \hat{c}_2(z) = \hat{y}_j(z), \quad \text{if } z \in Z_j$$

Multiplying both sides by $\hat{p}(z)$:

$$\hat{p}(z)\hat{c}_1(z) + \hat{p}(z)\hat{c}_2(z) = \hat{p}(z)\hat{y}_j(z), \quad \text{if } z \in Z_j$$

Plugging in equation **(1)** from our consumers problem for both countries yields:

$$\widehat{w}_1 L_1 + \widehat{w}_2 L_2 = \hat{p}(z) \hat{y}_j(z), \quad \text{if } z \in Z_j \quad (4)$$

From 2 and the production function we have that

$$\hat{p}(z) \hat{y}_j(z) = \widehat{w}_j a_j(z) \hat{y}_j(z) = \widehat{w}_j a_j(z) \frac{\hat{l}_j(z)}{a_j(z)} = \widehat{w}_j \hat{l}_j(z), \quad \text{if } z \in Z_j$$

Therefore plugging the above into (4) yields:

$$\widehat{w}_1 L_1 + \widehat{w}_2 L_2 = \widehat{w}_j \hat{l}_j(z), \quad \text{if } z \in Z_j$$

Now for each country we can integrate over what it produces:

$$\int_{z \in Z_j} (\widehat{w}_1 L_1 + \widehat{w}_2 L_2) dz = \int_{z \in Z_j} \widehat{w}_j \hat{l}_j(z) dz$$

Pulling the constants out of the integrals:

$$(\widehat{w}_1 L_1 + \widehat{w}_2 L_2) \int_{z \in Z_j} dz = \widehat{w}_j \int_{z \in Z_j} \hat{l}_j(z) dz$$

Plugging in labor market clearing:

$$(\widehat{w}_1 L_1 + \widehat{w}_2 L_2) \int_{z \in Z_j} dz = \widehat{w}_j L_j, \quad (5)$$

And now note that, since $z \in [0,1]$, we have:

$$\int_{z \in Z_j} dz = \begin{cases} \bar{z}, & \text{if } j = 1 \\ (1 - \bar{z}), & \text{if } j = 2 \end{cases}$$

Therefore plugging that into (5) we have

$$\begin{aligned} \bar{z}(\widehat{w}_1 L_1 + \widehat{w}_2 L_2) &= \widehat{w}_1 L_1 \\ (1 - \bar{z})(\widehat{w}_1 L_1 + \widehat{w}_2 L_2) &= \widehat{w}_2 L_2 \end{aligned}$$

Therefore using the above two equations and rearranging we get:

$$\frac{\widehat{w}_1}{\widehat{w}_2} = \frac{\bar{z} L_2}{(1 - \bar{z}) L_1}, \quad (6)$$

Now combining (6) with (3), and plugging in $L_1 = L_2 = L$ yields:

$$\frac{\bar{z}}{1 - \bar{z}} = e^{a - 2a\bar{z}}$$

Which means that $\bar{z} = \frac{1}{2}$. We could have exploited symmetry to arrive at this conclusion immediately, but here we learned how to derive it without relying on symmetry.

Equilibrium Allocations

We can normalize $\hat{w}_1 = 1$, and since $\bar{z} = \frac{1}{2}$ that implies that $\frac{\hat{w}_1}{\hat{w}_2} = 1$ and therefore $\hat{w}_2 = 1$

Plugging wages into (2) yields prices:

$$\hat{p}(z) = \begin{cases} e^{az}, & z \in \left[0, \frac{1}{2}\right] \\ e^{a(1-z)}, & z \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Which we can combine with (1) to get quantities consumed:

$$\hat{c}_j(z) = \begin{cases} \frac{L}{e^{az}}, & z \in [0, .5] \\ \frac{L}{e^{a(1-z)}}, & z \in [.5, 0] \end{cases}, \quad j = 1, 2$$

And then goods market clearing implies:

$$\hat{y}_1(z) = \begin{cases} \frac{2L}{e^{az}}, & z \in [0, .5] \\ 0, & z \in [.5, 0] \end{cases}$$

$$\hat{y}_2(z) = \begin{cases} 0, & z \in [0, .5] \\ \frac{2L}{e^{a(1-z)}}, & z \in [.5, 0] \end{cases}$$

And using the production technology gives:

$$\hat{l}_1(z) = \begin{cases} 2L, & z \in [0, .5] \\ 0, & z \in [.5, 0] \end{cases}$$

$$\hat{l}_2(z) = \begin{cases} 0, & z \in [0, .5] \\ 2L, & z \in [.5, 0] \end{cases}$$

DFS With Tariffs

Now suppose that each country imposes an ad valorem tariff $\tau - 1$ on imports from the other country.

Equilibrium Definition (sloppier with notation this time)

An equilibrium for this economy is prices $\{\hat{p}_j(z), \hat{w}_j\}_{j=1,2; z \in [0,1]}$ and allocations $\{\hat{c}_j(z), \hat{y}_j(z), \hat{l}_j(z)\}_{j=1,2; z \in [0,1]}$ such that **1, 2, 3** hold below

1. Given prices the consumer solves:

$$\max \int_0^1 \log c_j(z) dz$$

subject to their budget constraint and non-negative consumption (T_j is revenue from tariffs)

$$\int_0^1 p_j(z) c_j(z) dz = w_j L + T_j$$

$$c_j(z) \geq 0 \quad \forall z \in [0,1]$$

2. Firms maximize profits in each country $j = 1,2$ for each good z :

$$\max p_j^i(z) y_j^i(z) - w_j l_j^i(z)$$

Where (y_j^i is output produced in country j for consumption in country i , and $\tau - 1$ is the tariff)

$$y_j^i(z) = \frac{l_j^i(z)}{a_j(z)}$$

And countries have to charge a tariff on their exports to the other country

$$p_j^i(z) = \tau p_j^j, \quad i \neq j$$

3. Markets clear, $\forall z \in [0,1]$; $j = 1,2$:

$$c_j(z) = y_1^j(z) + y_2^j(z)$$

$$\int_0^1 (l_j^1(z) + l_j^2(z)) = L$$

And tariffs are transferred back to the consumer through T_j , where T_j is defined as:

$$T_j = \int_{[0,1] \setminus Z_j} (\tau - 1) p_i(z) y_i^j(z) dz, \quad i \neq j$$

Here Z_j is the set of goods country j produces (and so $[0,1] \setminus Z_j$ is the set they import)

Solving for the Equilibrium

Assume that tariffs aren't so high that the countries stop trading with each other: $\tau \leq \min \left\{ \frac{w_2}{w_1} e^a, \frac{w_1}{w_2} e^a \right\}$

Now we know there will be two cutoff goods: \bar{z}_1, \bar{z}_2 ; $\bar{z}_1 \leq \bar{z}_2$ such that the set of goods produced by each country is:

$$Z_1 = [0, \bar{z}_2]$$

$$Z_2 = [\bar{z}_1, 1]$$

And the set of goods exported by each country is

$$Z_1^2 = [0, \bar{z}_1]$$

$$Z_2^1 = [\bar{z}_2, 1]$$

Solving Consumers Problem:

The consumer's problem is essentially the same, except now he gets revenues from tariffs therefore something similar to equation (1) holds:

$$c_j(z) = \frac{w_j L + T_j}{p_j(z)} \quad \forall z \in [0, 1], \quad (7)$$

Solving Firms Problem:

The firm's problem is the same as in the free trade model, except now τ is applied to the price for exports. Therefore from the FOC for the firms problem we have prices given by

$$p_j(z) = \min\{p_i^j, p_j^j\}$$

Where

$$p_i^j = \begin{cases} w_i a_i(z), & i = j \\ \tau w_i a_i(z), & i \neq j \end{cases}$$

Production and Exporting Cutoffs:

Now, instead of having one cutoff good where the effective cost of producing is equal in both countries as in eqn (3), we have two such cutoff goods:

$$\frac{w_1}{w_2} = \tau^{-1} e^{a-2a\bar{z}_2}, \quad (8)$$

$$\frac{w_1}{w_2} = \tau e^{a-2a\bar{z}_1}, \quad (9)$$

And trade must be balanced (exports=imports) which means that:

$$\int_0^{\bar{z}_1} p_1^2(z) c_2(z) dz = \int_{\bar{z}_2}^1 p_2^1(z) c_1(z) dz$$

And from (7) this means

$$\int_0^{\bar{z}_1} (w_2 L + T_2) dz = \int_{\bar{z}_2}^1 (w_1 L + T_1) dz$$

Therefore

$$(w_2L + T_2)\bar{z}_1 = (w_1L + T_1)(1 - \bar{z}_2)$$

And we can normalize $w_2 = 1$, giving us:

$$w_1 = \frac{L\bar{z}_1 + T_2\bar{z}_1 - T_1 + \bar{z}_2T_1}{L(1 - \bar{z}_2)}$$

Combining the above with equations (8) and (9) gives us three equations in three unknowns ($w_1, \bar{z}_1, \bar{z}_2$).

Solving these three equations yields:

$$\bar{z}_1 = \frac{1}{2} \left(1 - \frac{1}{a} \log \tau \right)$$

$$\bar{z}_2 = 1 - \frac{1}{2} \left(1 - \frac{1}{a} \log \tau \right)$$

and $w_1 = w_2 = 1$.

Equilibrium Allocations:

Plugging in our wages to get prices, means that equilibrium allocations will be given by

$$c_1(z) = \begin{cases} \frac{L + T_1}{e^{az}}, & z \in [0, \bar{z}_2] \\ \frac{L + T_1}{\tau e^{a(1-z)}}, & z \in (\bar{z}_2, 1] \end{cases}$$

$$c_2(z) = \begin{cases} \frac{L + T_2}{\tau e^{az}}, & z \in [0, \bar{z}_1] \\ \frac{L + T_2}{e^{a(1-z)}}, & z \in (\bar{z}_1, 1] \end{cases}$$

where

$$p_1(z) = \begin{cases} e^{az}, & z \in [0, \bar{z}_2] \\ \tau e^{a(1-z)}, & z \in (\bar{z}_2, 1] \end{cases}$$

$$p_2(z) = \begin{cases} \tau e^{az}, & z \in [0, \bar{z}_1] \\ e^{a(1-z)}, & z \in (\bar{z}_1, 1] \end{cases}$$

And tariff transfers can be computed by, for country 1

$$\begin{aligned} T_1 &= \int_{[\bar{z}_2, 1]} (\tau - 1)p_2(z)y_2^1(z)dz = \int_{[\bar{z}_2, 1]} (\tau - 1)p_2(z) \left(\frac{L + T_1}{p_1(z)} \right) dz = \int_{[\bar{z}_2, 1]} (\tau - 1)p_2(z) \left(\frac{L + T_1}{\tau p_2(z)} \right) dz \\ &= \int_{[\bar{z}_2, 1]} \left(\frac{\tau - 1}{\tau} \right) (L + T_1) dz = \left(\frac{\tau - 1}{\tau} \right) (L + T_1)(1 - \bar{z}_2) \end{aligned}$$

which yields

$$T_1 = \frac{L \left(\frac{\tau - 1}{\tau} \right) (1 - \bar{z}_2)}{1 - \left(\frac{\tau - 1}{\tau} \right) (1 - \bar{z}_2)}$$

And similarly, for country 2

$$T_2 = \int_{[0, \bar{z}_1]} \left(\frac{\tau - 1}{\tau} \right) (L + T_2) dz \Rightarrow T_2 = \frac{L \left(\frac{\tau - 1}{\tau} \right) \bar{z}_1}{1 - \left(\frac{\tau - 1}{\tau} \right) \bar{z}_1}$$

And note that $\bar{z}_1 = 1 - \bar{z}_2$, therefore $T_1 = T_2 = T$, as expected in a symmetric world. Plugging in cutoffs we have:

$$T = \frac{L(\tau - 1)(a - \log \tau)}{a(1 + \tau) + (\tau - 1) \log \tau}$$

Note that therefore total income is

$$L + T = \frac{L}{1 - \left(\frac{\tau - 1}{\tau} \right) \bar{z}_1} = \frac{2aL\tau}{a(1 + \tau) + (\tau - 1) \log \tau}$$

Note also that we can compute GDP as [Note: Technically we should use output for GDP and consumption for real consumption, but since we have a symmetric world and all output is consumed the two are the same]

$$\begin{aligned} GDP_1 &= \int_0^{\bar{z}_2} p_1(z) c_1(z) dz + \int_0^{\bar{z}_1} p_1^2(z) c_2(z) dz \\ &= \int_0^{\bar{z}_2} (L_1 + T_1) dz + \int_0^{\bar{z}_1} (L_2 + T_2) dz = L + T = \frac{2aL\tau}{a(1 + \tau) + (\tau - 1) \log \tau} \end{aligned}$$

We can also compute Constant Price GDP, for a base period in which $\tau = 1$, as

$$RGDP_1 = \int_0^{\bar{z}_2} \tilde{p}_1(z) c_1(z) dz + \int_0^{\bar{z}_1} \tilde{p}_1^2(z) c_2(z) dz$$

where the $\tilde{p}(z)$ are the prices from the base case of free trade.

And note that $\tilde{p}_1 = p_1$ for $z \in [0, \frac{1}{2}]$, and $\tilde{p}_1 = e^{a(1-z)} = \frac{e^{a(1-z)}}{e^{az}} p_1$, and $\tilde{p}_1^2 = p_1^2 / \tau$ therefore

$$RGDP_1 = \int_0^{\frac{1}{2}} p_1(z) c_1(z) dz + \int_{\frac{1}{2}}^{\bar{z}_2} \frac{e^{a(1-z)}}{e^{az}} p_1(z) c_1(z) dz + \int_0^{\bar{z}_1} \frac{1}{\tau} p_1^2(z) c_2(z) dz$$

And plugging in that $p_1(z) c_1(z) = L + T \forall z \in [0, \bar{z}_2]$ and $p_1^2(z) c_2(z) = L + T \forall z \in [0, \bar{z}_1]$:

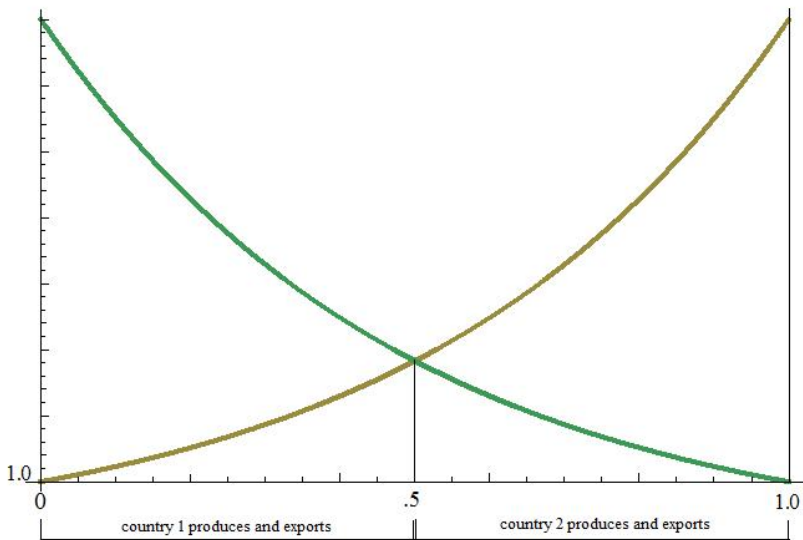
$$\begin{aligned}
RGDP_1 &= \int_0^{\frac{1}{2}} (L+T) dz + \int_{\frac{1}{2}}^{\bar{z}_2} \frac{e^{a(1-z)}}{e^{az}} (L+T) dz + \int_0^{\bar{z}_1} \frac{1}{\tau} (L+T) dz \\
&= (L+T) \int_0^{\frac{1}{2}} dz + (L+T) \int_{\frac{1}{2}}^{\bar{z}_2} \frac{e^{a(1-z)}}{e^{az}} dz + \frac{1}{\tau} (L+T) \int_0^{\bar{z}_1} dz \\
&= (L+T) \frac{1}{2} + (L+T) \left(\frac{1 - e^{a-2a\bar{z}_2}}{2a} \right) + \frac{1}{\tau} (L+T) \bar{z}_1 \\
&= GDP_1 \left(\frac{1}{2} + \left(\frac{1 - e^{a-2a\bar{z}_2}}{2a} \right) + \frac{1}{\tau} \bar{z}_1 \right)
\end{aligned}$$

Plugging in cutoffs

$$\begin{aligned}
RGDP_1 &= GDP_1 \left(\frac{1}{2} + \left(\frac{1 - e^{a-2a \left(1 - \frac{1}{2} \left(1 - \frac{1}{a} \log \tau \right) \right)}}{2a} \right) + \frac{1}{\tau} \frac{1}{2} \left(1 - \frac{1}{a} \log \tau \right) \right) \\
&= GDP_1 \left(\frac{1}{2} + \frac{1}{2} \left(\frac{\tau - 1}{a\tau} \right) + \frac{1}{\tau} \frac{1}{2} \left(1 - \frac{1}{a} \log \tau \right) \right) \\
&= \frac{GDP_1}{2} \left(\frac{a + \tau - 1 + a\tau - \log \tau}{a\tau} \right)
\end{aligned}$$

Which equals current price GDP if $\tau = 1$ and is less than current price GDP otherwise.

Free trade:



Tariffs/Iceberg Costs:

